The ratio f_K/f_π and its implication on the CKM matrix

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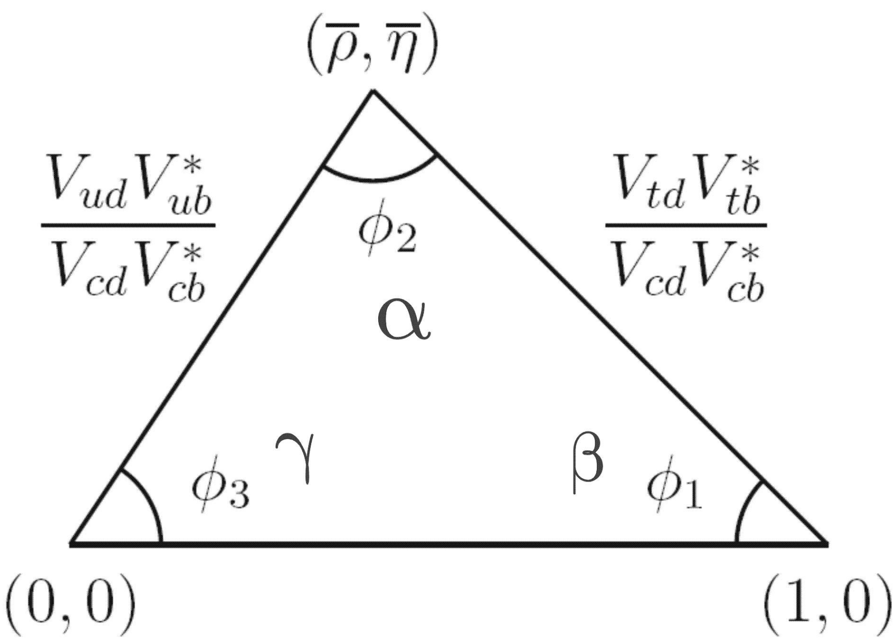
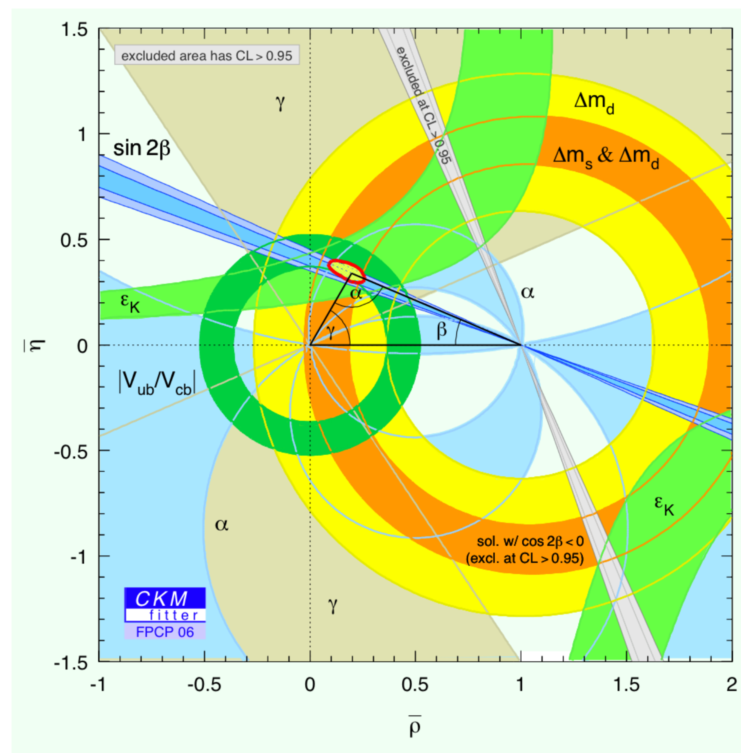
Overview (1): CKM matrix

ν_e	ν_μ	ν_τ
e	μ	τ
u	c	t
d	s	b

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$





Overview (2): Marciano's observation

W.J. Marciano, PRL 93 231803 (2004) [hep-ph/0402299]:

- $|V_{\rm ud}|$ is known, from super-allowed nuclear β-decays, with 0.03% precision [HT].

- $|V_{\rm us}|$ is much less precisely known, but can be linked to $|V_{\rm ud}|$ via a relation involving f_K/f_π, with everything else known rather accurately:

$$\frac{\Gamma(K\to l\bar{\nu}_l)}{\Gamma(\pi\to l\bar{\nu}_l)}=\frac{|V_{\rm us}|^2}{|V_{\rm ud}|^2}\frac{f_K^2}{f_\pi^2}\,\frac{M_K(1-m_l^2/M_K^2)^2}{M_\pi(1-m_l^2/M_\pi^2)^2}\{1+\frac{\alpha}{\pi}(C_K-C_\pi)\}$$

- CKM unitarity $|V_{\rm ud}|^2+|V_{\rm us}|^2+|V_{\rm ub}|^2=1$ (with $|V_{\rm ub}|$ being negligibly small) is genuine to the SM; any deviation is a *unambiguous* signal of BSM physics.

$\Longrightarrow$ calculate f_K/f_π in $N_f=2+1$ QCD (with quark masses extrapolated to the physical point) on the lattice; the precision attained gives the precision of $|V_{\rm us}|$.

Overview (3): analysis within/beyond Standard Model

$$|V_{\rm ud}|^2 + |V_{\rm us}|^2 + |V_{\rm ub}|^2 = 1 \qquad \text{[SM]}$$

$$|V_{\rm us}| f_+(0) = 0.21661(47) \qquad \text{[exp, FlavianetKaon 08]}$$

$$\frac{|V_{\rm us}| f_K}{|V_{\rm ud}| f_\pi} = 0.27599(59) \qquad \text{[exp, FlavianetKaon 08]}$$

$\longrightarrow$ 3 relations for 4 unknowns, since $|V_{\rm ub}| = 3.39(36)10^{-3}$ [PDG 08] is known/tiny

$\longrightarrow$ determine any one of $\underbrace{|V_{\rm ud}|,\ |V_{\rm us}|}_{\rm nucl/tau\ data},\ \underbrace{f_+(0),\ f_K/f_\pi}_{\rm lattice\ QCD}$ and get the remaining three

$\Longrightarrow$ abandon unitary constraint, get (almost) *model-independent* test of BSM physics

Overview (4): other information on $|V_{\rm ud}|$ and $|V_{\rm us}|$

- **super-allowed nuclear β-decays**

$|V_{\rm ud}|=0.97425(22)$ [HardyTowner 09]

$\longrightarrow$ $|V_{\rm us}|=0.22544(95),\ f_+(0)=0.9608(46),\ f_K/f_\pi=1.1927(59)$

- **decays of τ to hadrons$(S\!=\!1)+\nu$**

$|V_{\rm us}|=0.2165(26)(5)$ [Gamiz et al. 07]

$\longrightarrow$ $|V_{\rm ud}|=0.9763(6),\ f_+(0)=1.001(12),\ f_K/f_\pi=1.245(16)$

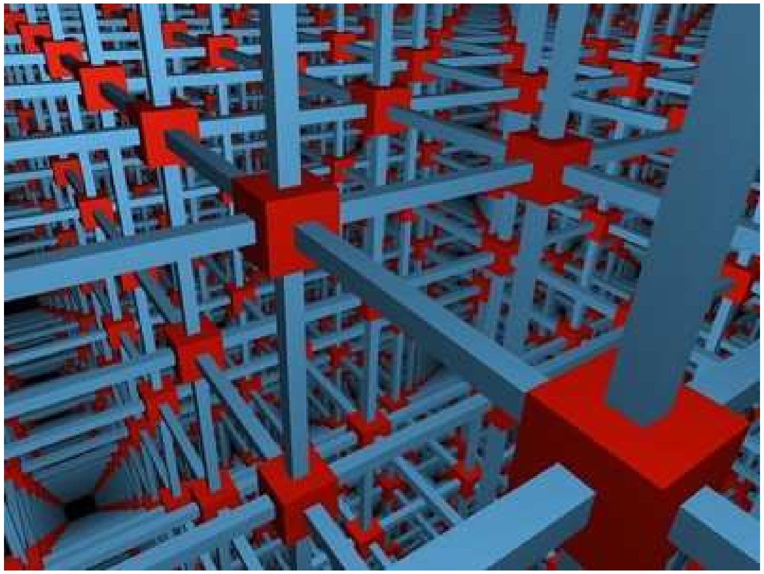
- **decays of τ to hadrons$(S\!=\!1)+\nu$ with $J_{\rm e.m.}$ constraint**

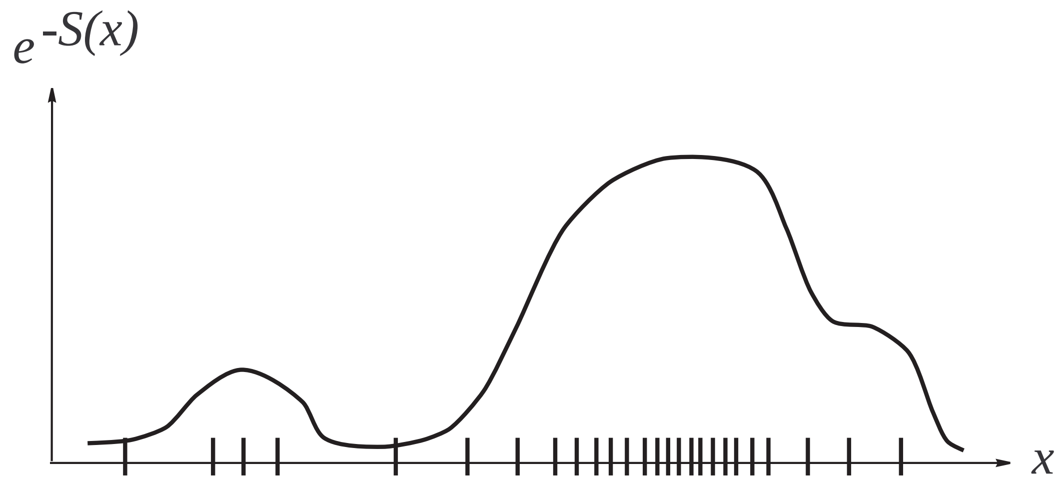
$|V_{\rm us}|=0.2208(39)$ [Maltman 09]

$\longrightarrow$ $|V_{\rm ud}|=0.9753(9),\ f_+(0)=0.981(17),\ f_K/f_\pi=1.219(23)$

$\Longrightarrow$ determine f_K/f_π in QCD to resolve the ambiguity and test the SM

Lattice QCD basics (1): path-integral quantization





- Define space-time as regular 4D grid (spacing a) with periodic boundary conditions.
- Put matter fields on sites: scalar $\phi(x)$ or spinor $\psi(x)$ with $x=(an_1,..,an_4)$.
- Put gauge fields on links: photon or gluon within $U_\mu(x)=\exp(\mathrm{i}\int_x^{x+\hat{\mu}} A_\mu(x')dx')$.
- Define gluon and fermion action with correct weak-coupling limit and $S=S_G+S_F$.
- Define $Z=\int DU D\bar{\psi} D\psi \exp(-S[U,\bar{\psi},\psi])$ via integration over *all* field variables.
- Use methods from statistical mechanics to sample *relevant* field configurations.

Lattice QCD basics (2): discretization

- **QFT on the lattice**

  $Z = \int D\phi\, e^{-S[\phi]}$, $S[\phi] = \frac{1}{2}(\nabla\phi)^2 + \frac{m}{2}\phi^2 + \ldots$, $D\phi$ means $-\infty < \phi(x) < \infty$ for each x

- **Gluons on the lattice**

  $U_\mu(x)U_\nu(x+a\hat{\mu}) - U_\nu(x)U_\mu(x+a\hat{\nu}) = \mathrm{i}a^2 F_{\mu\nu}(x) + O(a^3)$

  $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + ig[A_\mu(x), A_\nu(x)]$

  $S[U] = \beta \sum_\Box \{1 - \frac{1}{3}\mathrm{ReTr}(U_\Box)\} \;\rightarrow\; \frac{a^4}{g^2} \sum_{x,\mu<\nu} \mathrm{Tr}(F_{\mu\nu}(x)^2)$

- **Quarks on the lattice**

  $S[U] \;\rightarrow\; S[U] - \log(\det(D[U]))$, still integrate over $SU(3)$ for each link

- **Simulation setup**

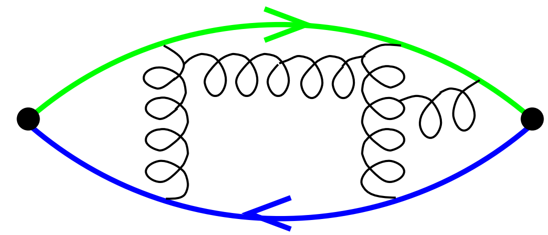
  1. generate configurations U distributed according to $p[U] = e^{-S[U]} \det^{N_f}(D[U])$
  2. solve $D[U]x = b$, build propagators to measure $C(t)$ for various states
  3. use lattice perturbation theory to renormalize/match to continuum schemes
  4. use effective field theories to extrapolate $a \rightarrow 0$, $L \rightarrow \infty$, $m_q \rightarrow m_q^{\mathrm{phys}}$

Lattice QCD spectroscopy (1): propagators

Hadronic correlator in $N_f \geq 2$ QCD: $C(t) = \int d^4x\, C(t,\mathbf{x})\, e^{i\mathbf{px}}$ with

$$C(x) = \langle O(x)\, O(0)^\dagger \rangle = \frac{1}{Z} \int DU\, D\bar{q} Dq\;\; O(x)\, O(0)^\dagger\;\; e^{-S_G - S_F}$$

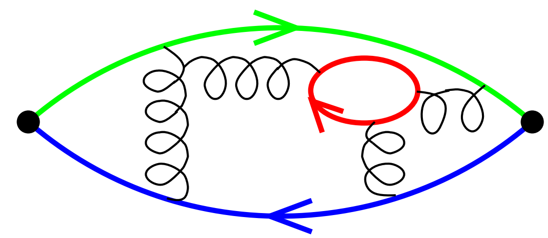
where $O(x) = \bar{d}(x)\Gamma u(x)$ and $\Gamma = \gamma_5, \gamma_4\gamma_5$ for $\pi^\pm$ and
$S_G = \beta \sum (1 - \frac{1}{3}\mathrm{Re}\mathrm{Tr}\, U_{\mu\nu}(x))$, $S_F = \sum \bar{q}(D+m)q$



(A) Quenched QCD: quark loops neglected

$$\langle \bar{d}(x)\Gamma_1 u(x)\; \bar{u}(0)\Gamma_2 d(0) \rangle = \frac{1}{Z} \int DU\;\; \det(D+m)^{N_f}\; e^{-S_G}$$

$$\times\; \mathrm{Tr}\Big\{ \Gamma_1 (D+m)^{-1}_{x0}\, \Gamma_2\, \underbrace{(D+m)^{-1}_{0x}}_{\gamma_5 [(D+m)^{-1}_{x0}]^\dagger \gamma_5} \Big\}$$

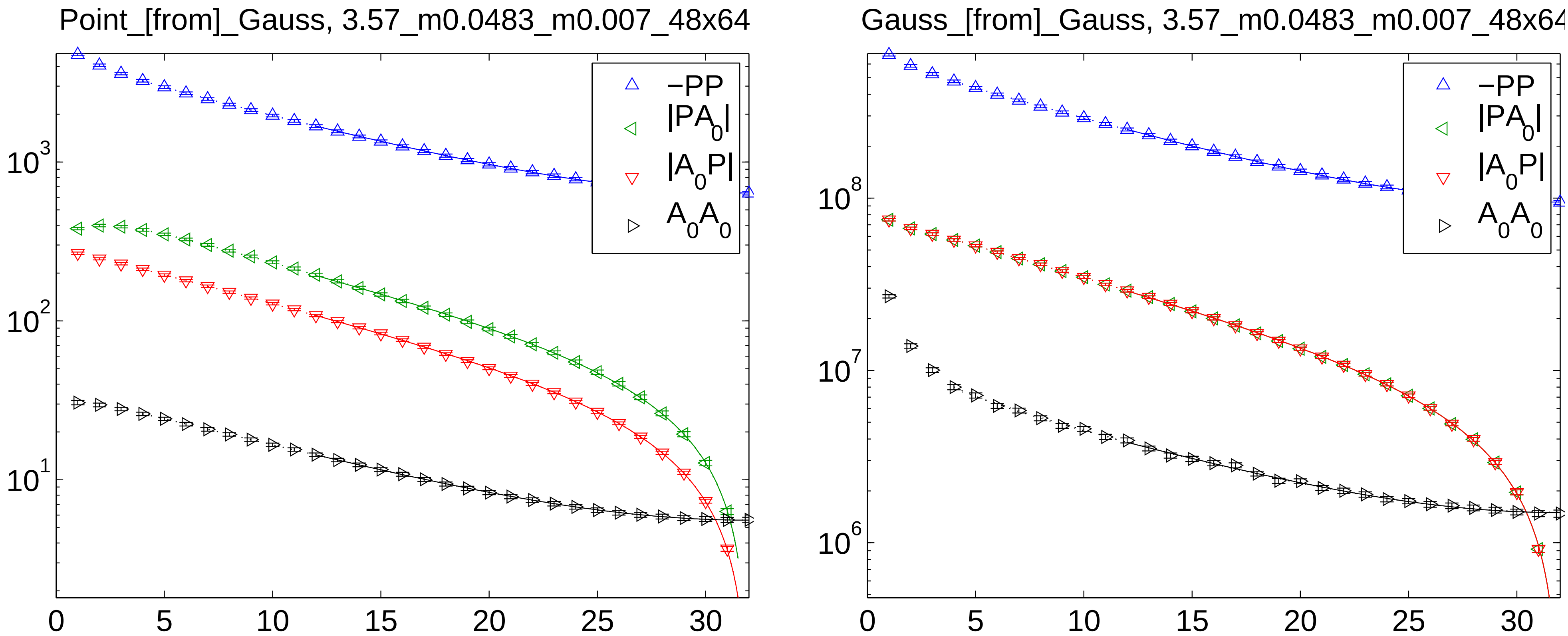


(B) Full QCD

- Choose $m_u = m_d$ to save CPU time, since isospin $SU(2)$ is a good symmetry.
- In principle $m_{\mathrm{valence}} = m_{\mathrm{sea}}$, but often additional valence quark masses to broaden data base. Note that "partially quenched QCD" is an *extension* of "full QCD".
- $(D+m)^{-1}_{x0}$ for all x amounts to 12 *columns* (with spinor and color) of the inverse.

Lattice QCD spectroscopy (2): correlators

Excellent data quality even on our lightest ensemble ($M_\pi \simeq 190\,\mathrm{MeV}$ and $L \simeq 4.0\,\mathrm{fm}$):



$\cosh(.)/\sinh(.)$ for $-PP, |PA_0|, |A_0P|, A_0A_0$ with Gauss source and local/Gauss sink

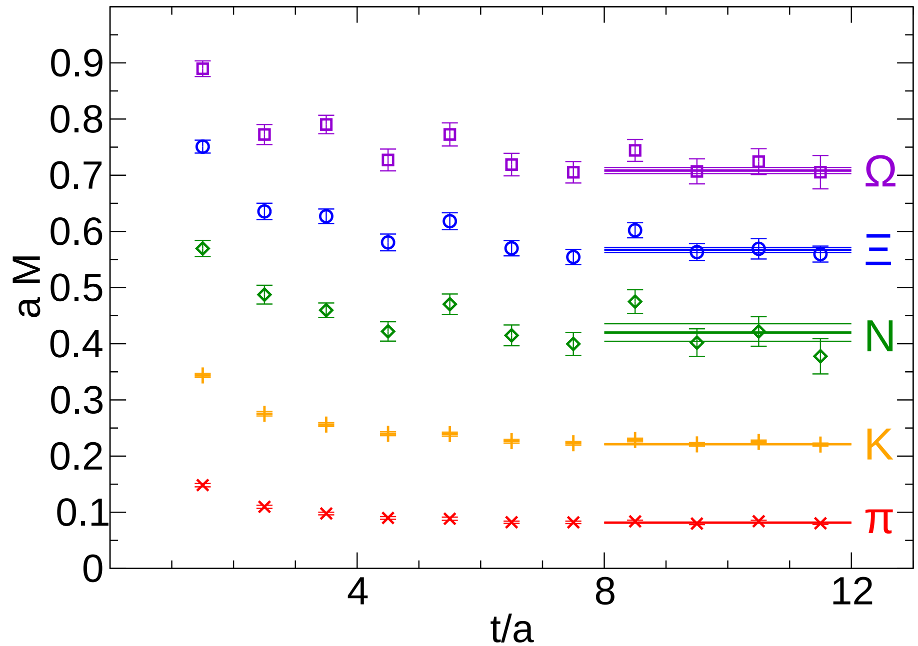
$C_{Xx,Yy}(t) = c_0 e^{-M_0 t} \pm c_0 e^{-M_0(T-t)} + \ldots$ with $X, Y \in \{P, A_0\}$ and $x, y \in \{\mathrm{loc}, \mathrm{gau}\}$

$\longrightarrow$ $c_0 = G\tilde{G}/M_0, G\tilde{F}, F\tilde{G}, F\tilde{F}M_0$ (left) and $c_0 = \tilde{G}\tilde{G}/M_0, \tilde{G}\tilde{F}, \tilde{F}\tilde{G}, \tilde{F}\tilde{F}M_0$ (right)

$\longrightarrow$ combined 1-state fit of 8 correlators with 5 parameters yields $M_\pi, F_\pi, m_{\mathrm{PCAC}}$

Lattice QCD spectroscopy (3): cost growth

With similar techniques for other channels we find in each run
aM_π, aM_K, aM_ρ, aM_{K^*}, aM_N, aM_Σ, aM_Ξ, aM_Λ, aM_Δ, aM_{Σ^*}, aM_{Ξ^*}, aM_Ω.



Cost growth (Lattice 2001, "Berlin wall phenomenon") recently tamed [in two parts]:

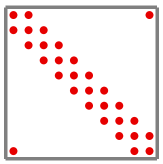
$a \to 0$	"continuum limit"	cost $\propto (1/a)^{4-6}$
$V \to \infty$	"infinite volume limit"	cost $\propto V^{5/4}$ with HMC
$m_{ud} \to m_{ud}^{\text{phys}}$	"chiral limit"	cost $\propto (1/m)^{1-2}$ with tricks
δ(observable) $\to 0$	"reduce statistical error"	cost $\propto \delta^{-2}$

Technicalities (1): sparse matrix inversion

$$D_{\rm st}(x,y) = \frac{1}{2}\sum_\mu \eta_\mu(x)\Big\{U_\mu(x)\delta_{x+\hat\mu,y} - U_\mu^\dagger(x-\hat\mu)\delta_{x-\hat\mu,y}\Big\} + m\delta_{x,y}$$

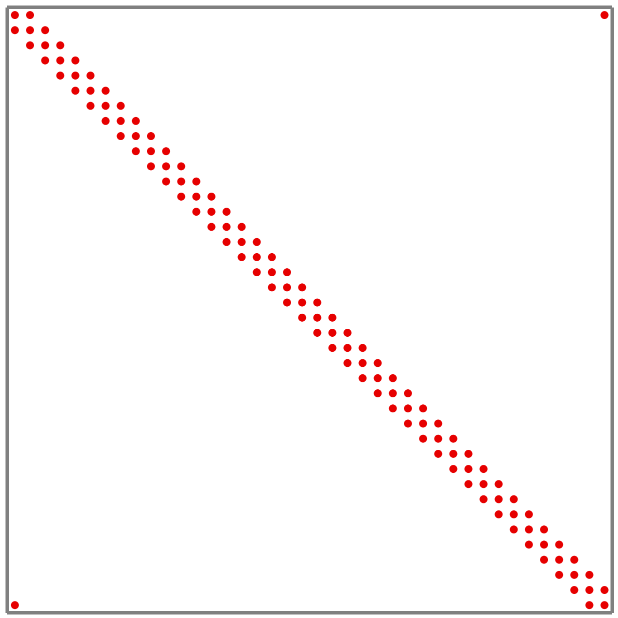
$$D_{\rm W}(x,y) = \frac{1}{2}\sum_\mu \Big\{(\gamma_\mu - I)U_\mu(x)\delta_{x+\hat\mu,y} - (\gamma_\mu + I)U_\mu^\dagger(x-\hat\mu)\delta_{x-\hat\mu,y}\Big\} + (4+m_0)\delta_{x,y}$$

staggered:

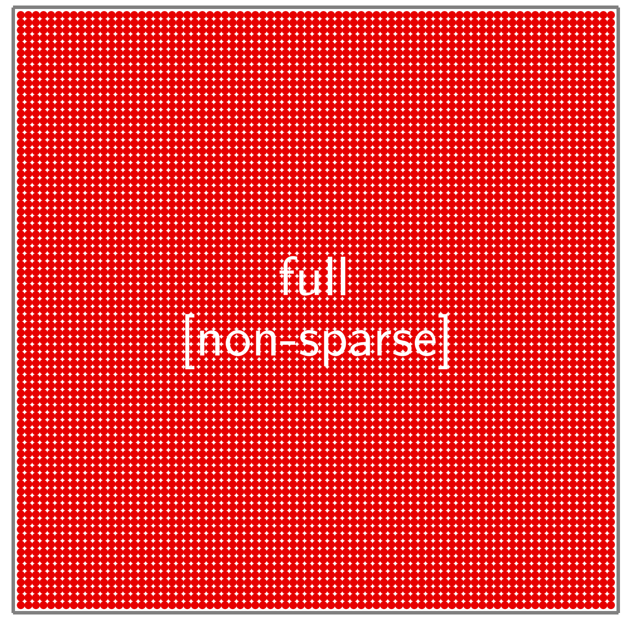


$\eta_\mu(x) = (-)^{\sum_{\nu<\mu} x_\nu}$

Wilson:



overlap:



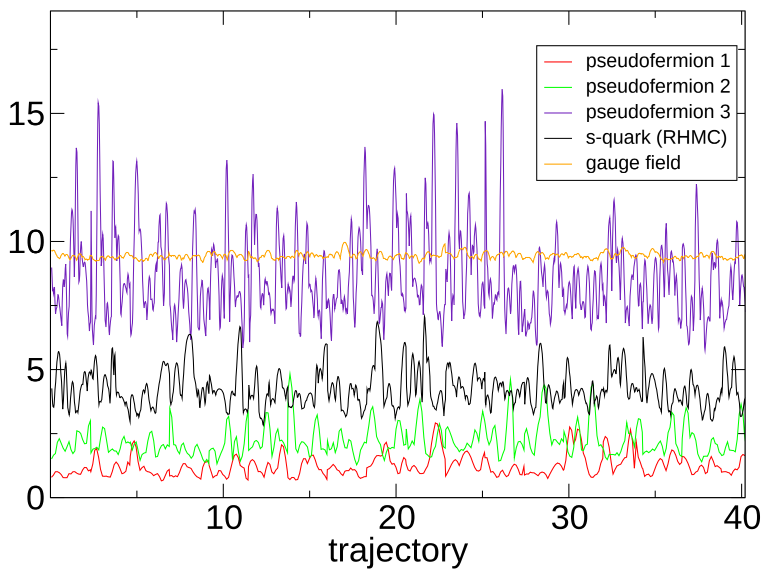
- Wilson: $D \equiv D\!\!\!/$ is $12N\times 12N$ complex sparse matrix, since (in chiral representation) any line/column contains only $3\cdot(1+2\cdot 8) = 51$ non-zero entries.
- Any inverse is full [non-sparse].
- CG solver yields $D^{-1}\eta \simeq c_0\eta + c_1 D\eta + ... + c_n D^n\eta$ with $n^2 \propto \mathrm{cond}(D^\dagger D) = \frac{\lambda_{\max}}{\lambda_{\min}}$.

Technicalities (2): stochastic determinant evaluation

Full QCD requires (frequent) evaluations of $\det(D)$, but:

- state-of-the-art lattices have $L/a = 64$ and thus $N = 64^3 \cdot 128 = 33'554'432$ sites
- D for Wilson-like fermions is $12N \times 12N = 402'653'184 \times 402'653'184$ matrix
- storing $16 \cdot 10^{16}$ complex numbers in single precision takes $128 \cdot 10^{16}$ bytes
- complete 72-rack BG/P at Jülich has $144\,\mathrm{TB}$ memory, i.e. $144 \cdot 10^{12}$ bytes

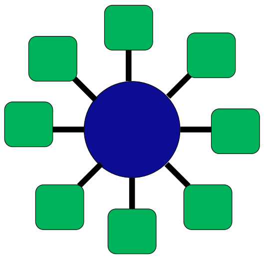
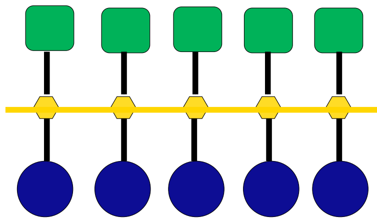
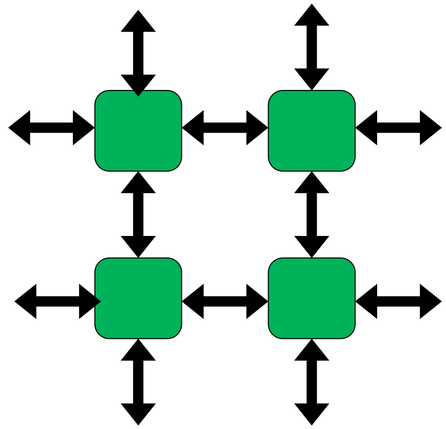
$N_f = 2$ part: $$\det^2(D) = \det(D^\dagger D) = \frac{1}{\det((D^\dagger D)^{-1})} = \int D\phi^\dagger D\phi \; e^{-\phi^\dagger (D^\dagger D)^{-1} \phi}$$



BMW uses battery of tricks:

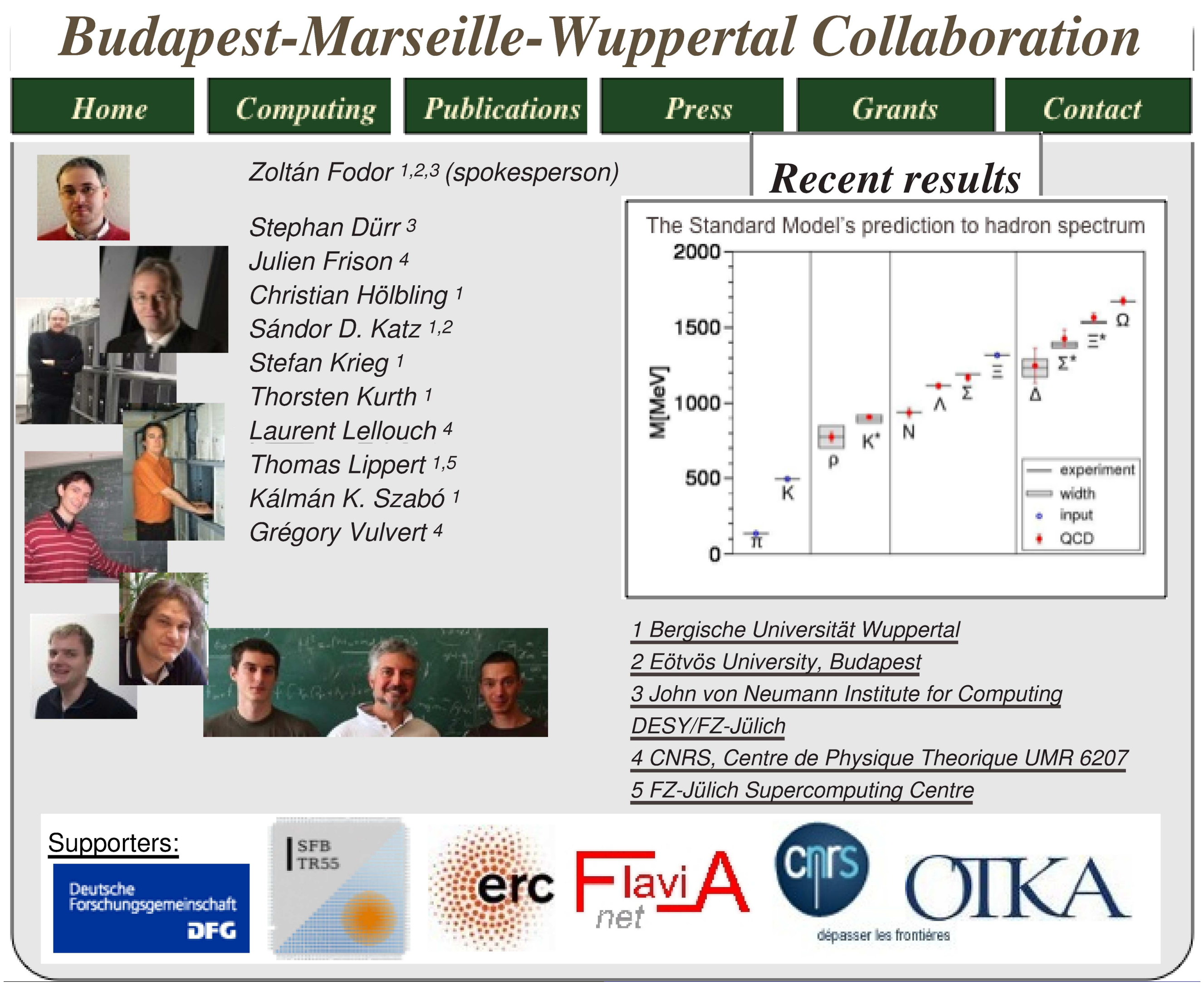
- even-odd preconditioning
- multiple time-scale integration ("Sexton-Weingarten scheme")
- mass preconditioning ("Hasenbusch trick")
- Omelyan integrator
- RHMC acceleration with multiple pseudofermions
- mixed-precision solver
- direct SPI (as opposed to MPI) implementation: 37% sustained performance and perfect weak scaling [problem size grows] up to full 72 racks

Technicalities (3): machine details


"JUGENE" [IBM BG/P]	02/2008 - 02/2009	06/2009 - ...
processor type	32-bit PowerPC 450 core 850 MHz	(3.4 Gflops each)
compute node	4-way SMP processor	
racks, nodes, processors	16, 16'384, 65'536	72, 73'728, 294'912
memory	2 GB per node, aggregate 32 TB	aggregate 144 TB
performance (peak/Lapack)	223/180 Teraflops [double prec.]	1/0.825 Petaflops
power consumption	$<$40 kW/rack, aggregate 0.5 MW	2.2 Megawatt
network topology	3D torus among compute nodes (plus global tree collective network, plus ethernet admin network)	
network latency	160 nsec (light travels 48 meters)	
network bandwidth	5.1 Gigabyte/s	

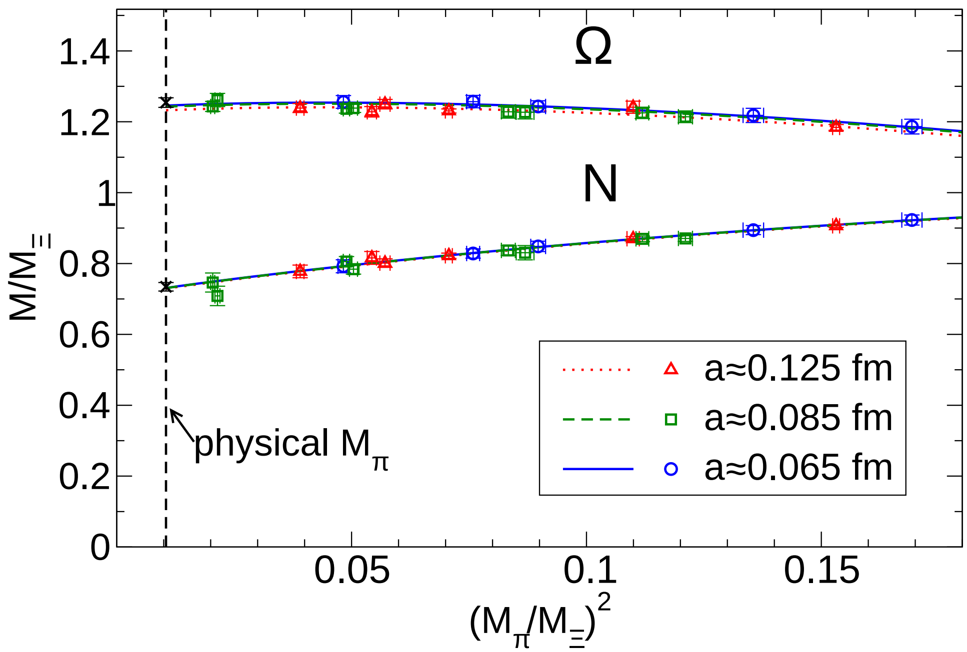
Budapest-Marseille-Wuppertal Collaboration

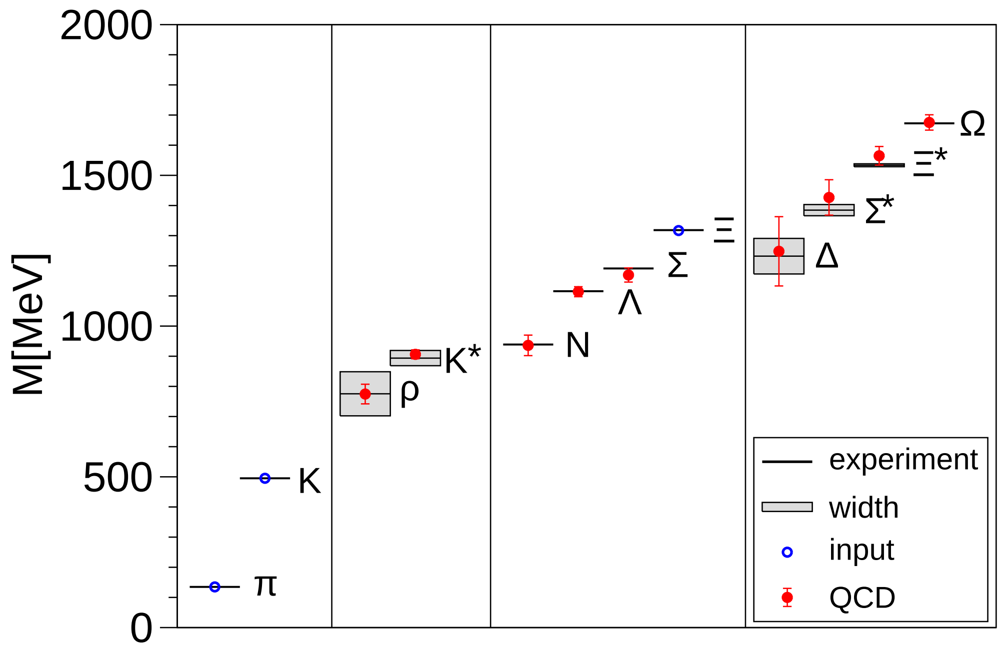


S. Dürr, Z. Fodor (spokesperson), C. Hoelbling, S.D. Katz, S. Krieg, T. Kurth, L. Lellouch, T. Lippert, A. Ramos, K.K. Szabo, G. Vulvert

f_K/f_π calculation (1): scale setting

BMW sets scale through M_Ω and/or M_Ξ [S. Dürr et al, Science 322, 1224 (2008)]:

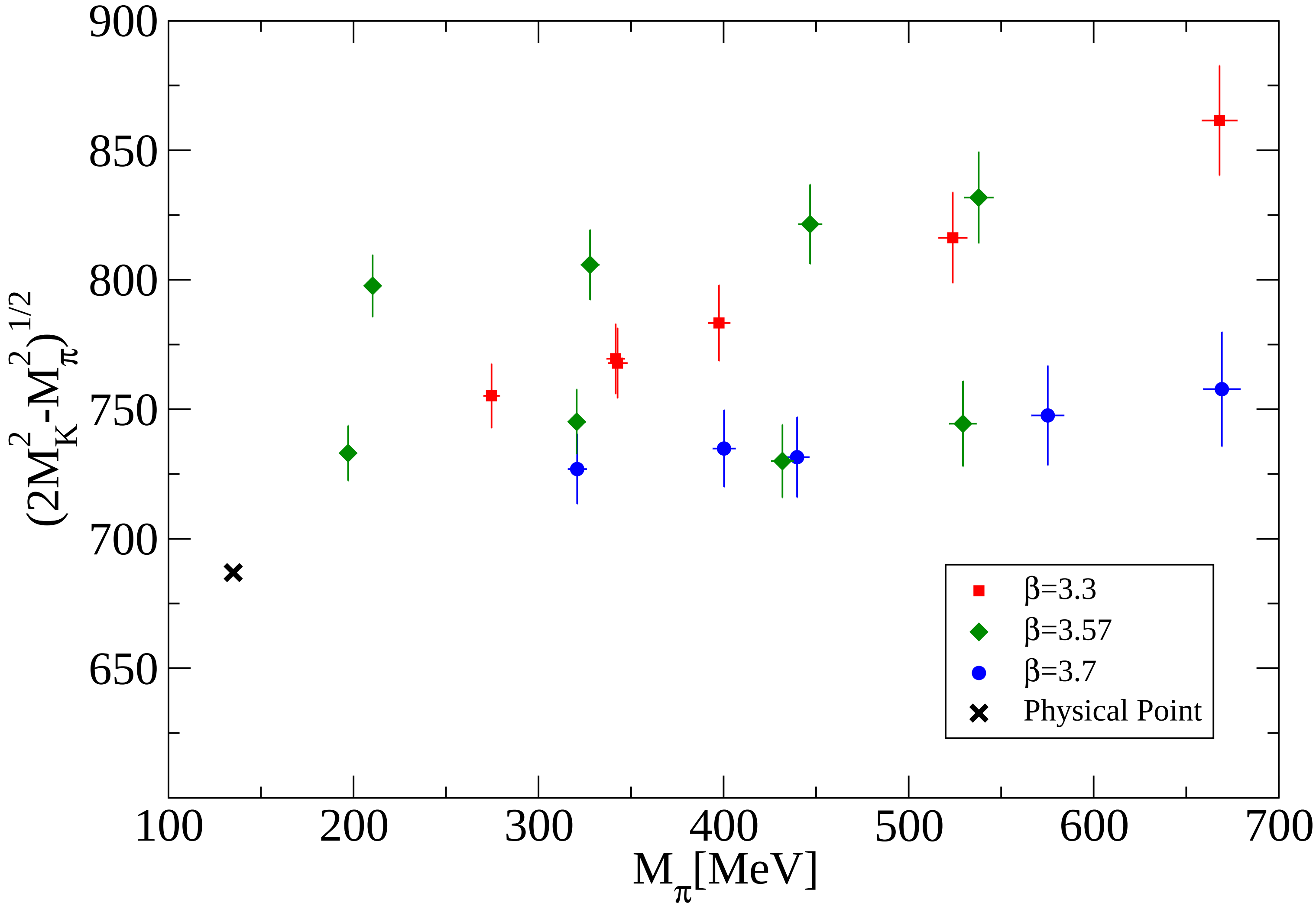




- large volumes ($M_\pi L \geq 4$ maintained, larger/smaller volumes for check)
- light pions ($M_\pi \simeq 190\,\mathrm{MeV}$ at one lattice spacing)
- three lattice spacings ($a \simeq 0.065, 0.085, 0.125\,\mathrm{fm}$)

f_K/f_π calculation (2): adjusting m_{ud}, m_s

$N_f = 2+1$ lattice QCD: set m_{ud}, m_s by adjusting M_π, M_K to their physical values



$\longrightarrow$ extract f_K/f_π on unitary ensembles and extrapolate to the physical mass point

$\longrightarrow$ $f_K/f_\pi = 1$ at $m_{ud} = m_s$ means that $f_K/f_\pi - 1$ is calculated with $\sim 5\%$ accuracy

f_K/f_π calculation (3): chiral extrapolation

- chiral $SU(3)$ formula:

$$\frac{F_K}{F_\pi} = 1+\frac{1}{32\pi^2F_0^2}\Big\{\frac{5}{4}M_\pi^2\log(\frac{M_\pi^2}{\mu^2})-\frac{1}{2}M_K^2\log(\frac{M_K^2}{\mu^2}) -[M_K^2-\frac{1}{4}M_\pi^2]\log(\frac{4M_K^2-M_\pi^2}{3\mu^2})\Big\}+\frac{4}{F_0^2}[M_K^2-M_\pi^2]L_5$$

- chiral $SU(2)$_plus_strange formula [RBC/UKQCD 08], simplified form:

$$\frac{F_K}{F_\pi}=\frac{F_K}{F_\pi}\Big|_{m_{ud}=0}\Big\{1+\frac{5}{8}\frac{M_\pi^2}{(4\pi F)^2}\log\Big(\frac{M_\pi^2}{\Lambda^2}\Big)\Big\}$$

- polynomial expansion $F_\pi/F_K=d_0+d_1(M_\pi-M_\pi^{\mathrm{ref}})+d_2(M_\pi-M_\pi^{\mathrm{ref}})^2$, e.g. around $M_\pi^{\mathrm{ref}}=300\,\mathrm{MeV}$, at fixed physical m_s, suggests [$\Delta_{\pi,K}=(M_{\pi,K}^2-M_{\pi,K}^{\mathrm{ref}\,2})/M_\Omega^2$]:

$$\frac{F_K}{F_\pi}=c_0+c_1\Delta_\pi+c_2\Delta_\pi^2+c_3\Delta_K$$

$\longrightarrow$ use all of them and treat spread as indicative of systematic uncertainty

f_K/f_π calculation (4): infinite volume extrapolation

- finite volume effects on F_K, F_π are known at the 2-loop level [CDH 05]

$$\frac{F_\pi(L)}{F_\pi} = 1 + \sum_{n=1}^{\infty} \frac{m(n)}{\sqrt{n}} \frac{1}{M_\pi L} \; 1 \; \frac{M_\pi^2}{(4\pi F_\pi)^2} \left[I_{F_\pi}^{(2)} + \frac{M_\pi^2}{(4\pi F_\pi)^2} I_{F_\pi}^{(4)} + \ldots \right]$$

$$\frac{F_K(L)}{F_K} = 1 + \sum_{n=1}^{\infty} \frac{m(n)}{\sqrt{n}} \frac{1}{M_\pi L} \frac{F_\pi}{F_K} \frac{M_\pi^2}{(4\pi F_\pi)^2} \left[I_{F_K}^{(2)} + \frac{M_K^2}{(4\pi F_\pi)^2} I_{F_K}^{(4)} + \ldots \right]$$

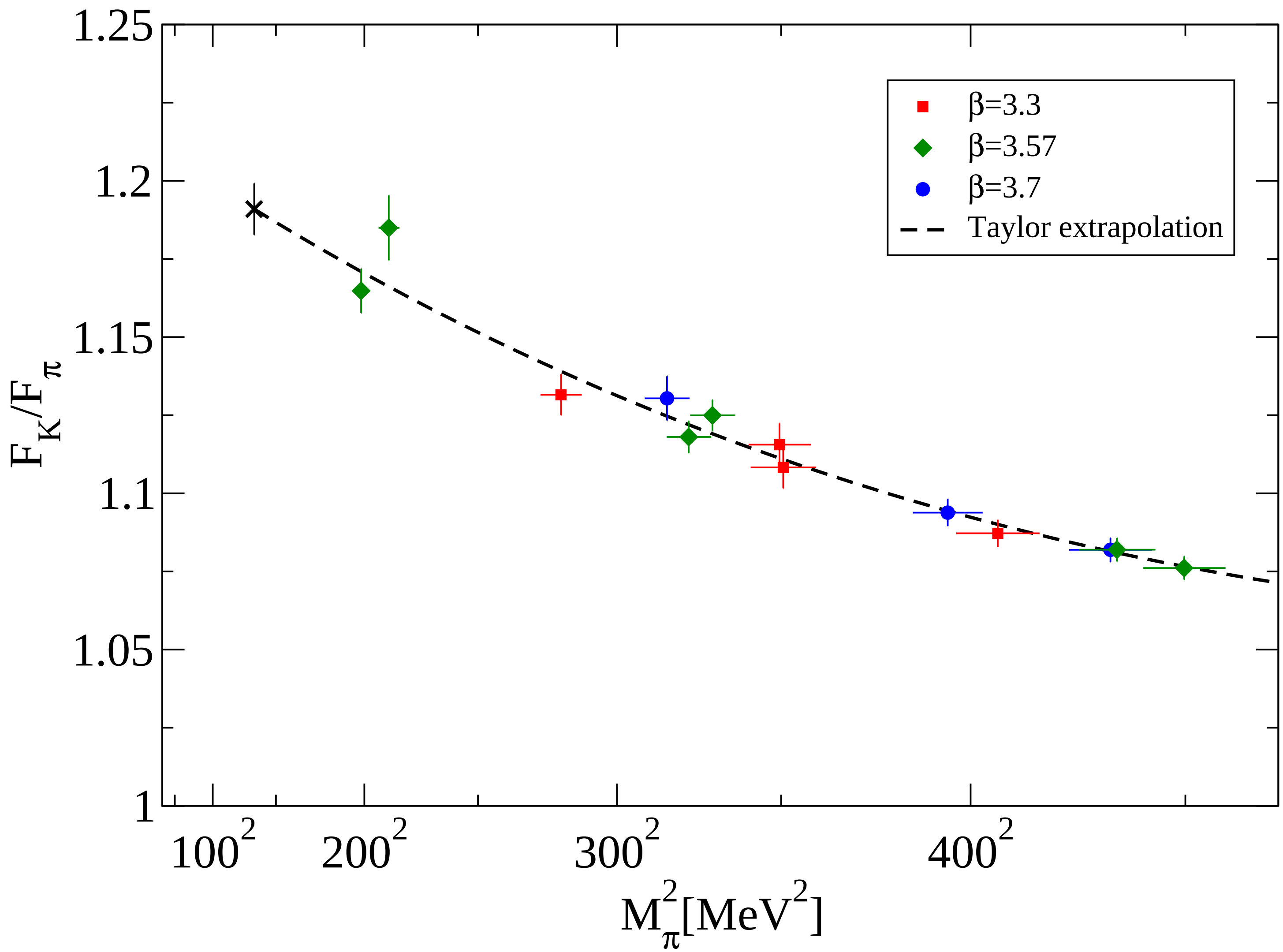
with $I_{F_\pi}^{(2)} = -4K_1(\sqrt{n}\,M_\pi L)$ and $I_{F_K}^{(2)} = -\frac{3}{2}K_1(\sqrt{n}\,M_\pi L)$, where $K_1(.)$ is a Bessel function of the second kind, and lengthy expressions for $I_{F_\pi}^{(4)}, I_{F_K}^{(4)}$

- finite volume effects cancel partly in the ratio, as evident from the 1-loop formula

$$\frac{F_K(L)}{F_\pi(L)} = \frac{F_K}{F_\pi} \left\{ 1 + \sum_{n=1}^{\infty} \frac{m(n)}{\sqrt{n}} \frac{1}{M_\pi L} \frac{M_\pi^2}{(4\pi F_\pi)^2} \left[\frac{F_\pi}{F_K} I_{F_K}^{(2)} - I_{F_\pi}^{(2)} \right] \right\}$$

- BMW uses $\frac{F_K(L)}{F_\pi(L)}/\frac{F_K}{F_\pi}$ at 1-loop and 2-loop level, and $F_\pi(L)/F_\pi$ at 2-loop level

f_K/f_π calculation (5): combined fits

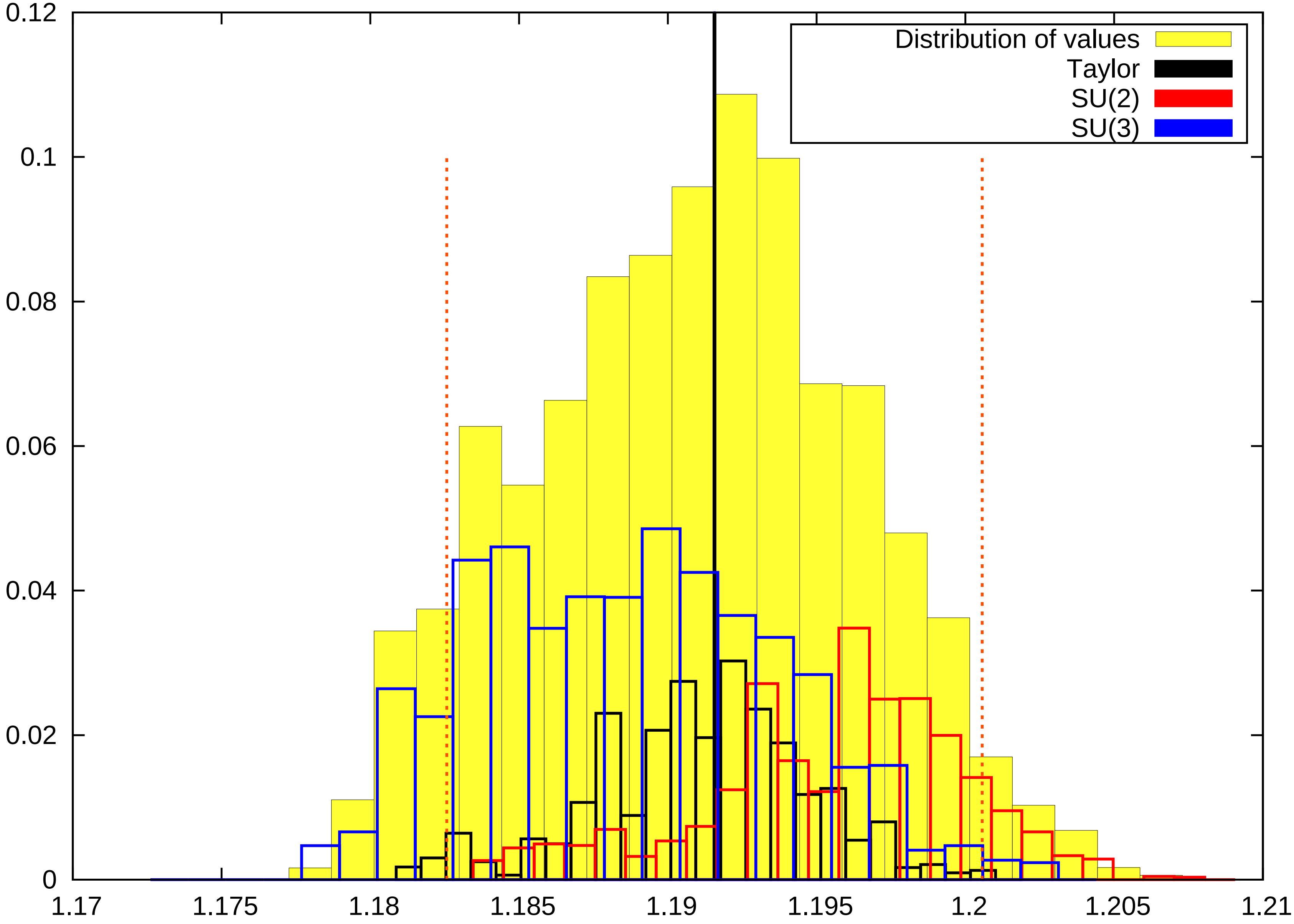


$\longrightarrow$ plot shows $\mathrm{data}(M_\pi^2, 2M_K^2{-}M_\pi^2) - \mathrm{fit}(M_\pi^2, 2M_K^2{-}M_\pi^2) + \mathrm{fit}(M_\pi^2, [2M_K^2{-}M_\pi^2]_{\mathrm{phys}})$

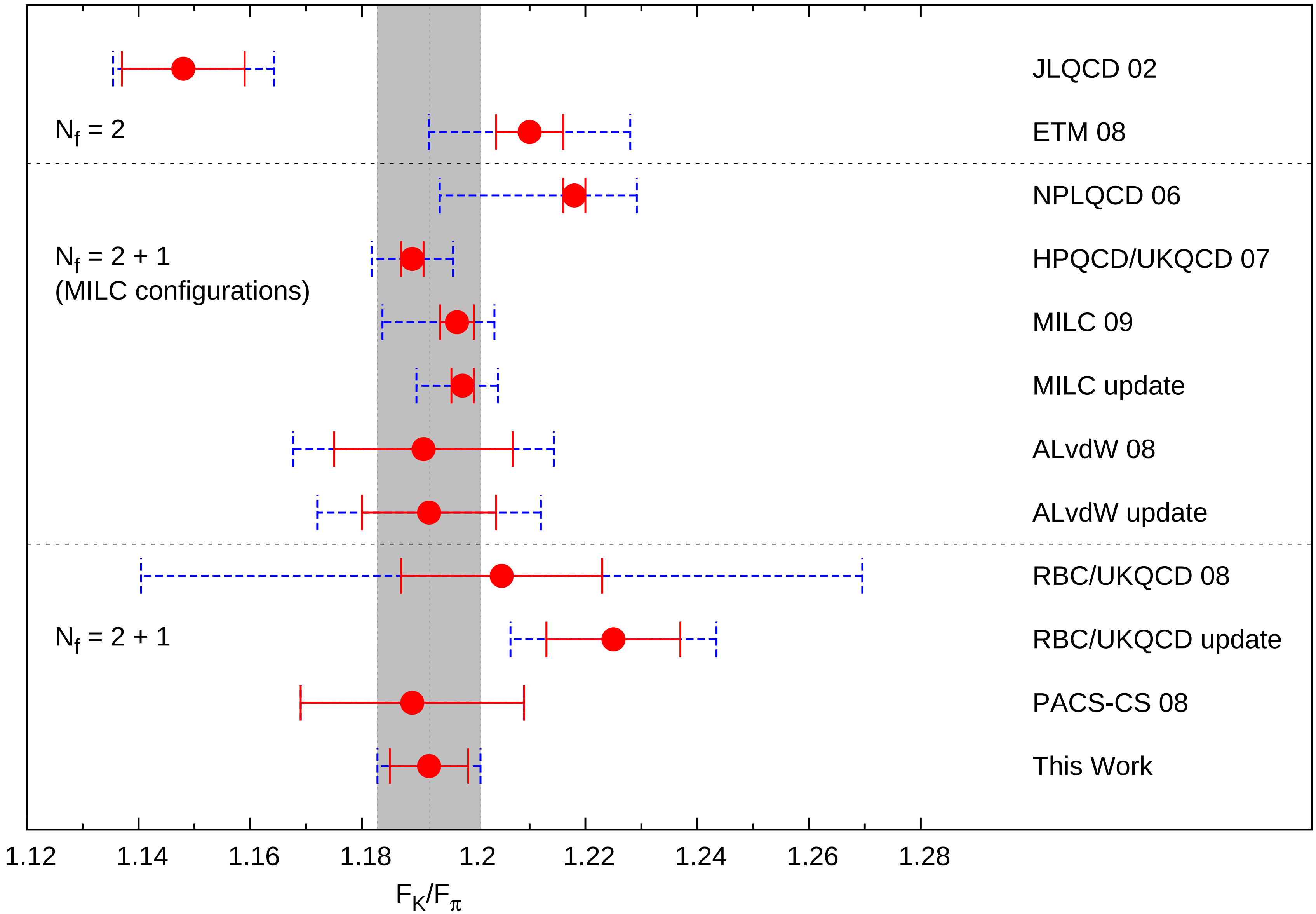
$\longrightarrow$ f_K/f_π scales rather nicely [note $a^2/\mathrm{fm}^2 = 0.0042, 0.0072, 0.0156$]

$\Longrightarrow$ $f_K/f_\pi = 1.192(7)(6)$ at physical m_{ud}, in continuum, in infinite volume

f_K/f_π calculation (6): error assessment



f_K/f_π calculation (7): comparison



f_K/f_π calculation (8): update on $|V_{\mathrm{us}}|$ and CKM unitarity

- Latest nuclear structure calculations [Hardy Towner'09] give

$$|V_{\mathrm{ud}}| = 0.97425(22)\ .$$

- Plug experimental information $\Gamma(K\to\mu\bar{\nu})/\Gamma(\pi\to\mu\bar{\nu}) = 1.3363(37)$ [PDG'08] and $C_K - C_\pi = -3.0 \pm 1.5$ [Marciano] into Marciano's equation; this yields

$$\frac{|V_{\mathrm{us}}|}{|V_{\mathrm{ud}}|}\frac{f_K}{f_\pi} = 0.27599(59)\ .$$

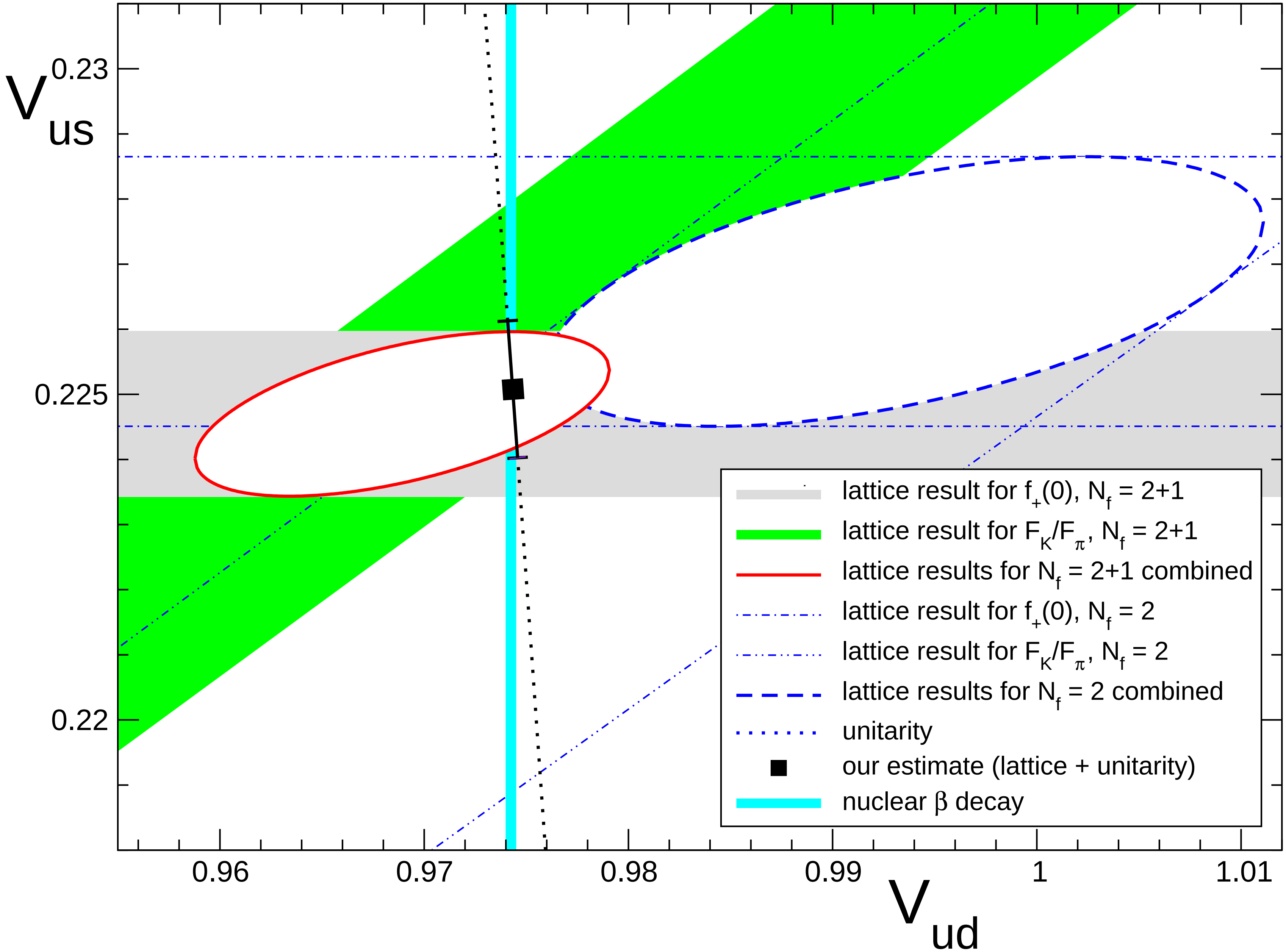
- Upon combining the previous one/two points and our value for f_K/f_π we obtain

$$|V_{\mathrm{us}}|/|V_{\mathrm{ud}}| = 0.2315(19) \quad \text{and} \quad |V_{\mathrm{us}}| = 0.2256(17)\ .$$

- Upon including $|V_{\mathrm{ub}}|=3.39(36)10^{-3}$ [PDG'08] we end up with [BMW, 1001.4692]

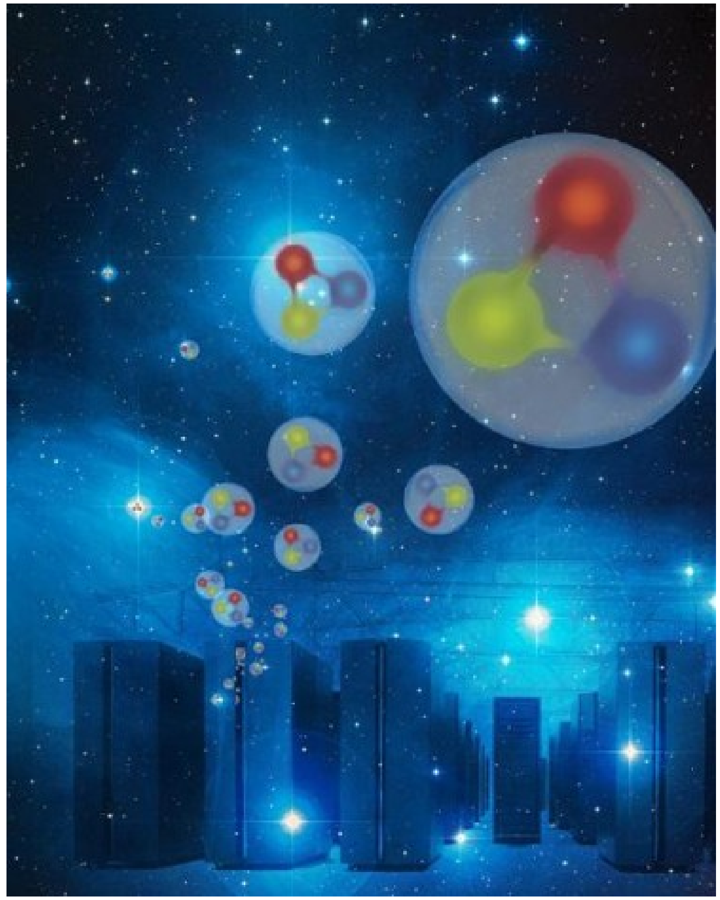
$$|V_{\mathrm{ud}}|^2 + |V_{\mathrm{us}}|^2 + |V_{\mathrm{ub}}|^2 = 1.0001(9)\ .$$

FLAG preview: information landscape



Summary

LQCD as a first-principles based approach for solving QCD has come of age:



locally known as "JUMP under water"

- quenched spectroscopy calculations since 20 years [GF-11 to CP-PACS]
- nowadays determinant of light quarks included [$M_\pi \simeq 140\,\mathrm{MeV}$ to come]
- all systematics controlled [excited states, $a \to 0$, $V \to \infty$, $m_q \to m_q^{\mathrm{phys}}$]
- important physics applications: f_K/f_π, f_{D_s}/f_D, B_K, $\langle N|\bar{u}u+\bar{d}d|N\rangle$, ...
- hard problems remain: $\Delta I = 1/2$, ϵ'/ϵ, resonances, ...