Chiral perturbation theory for lattice practitioners

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Challenge: Solving QCD ab initio

high energy frontier

LHC/B-factory/... observable

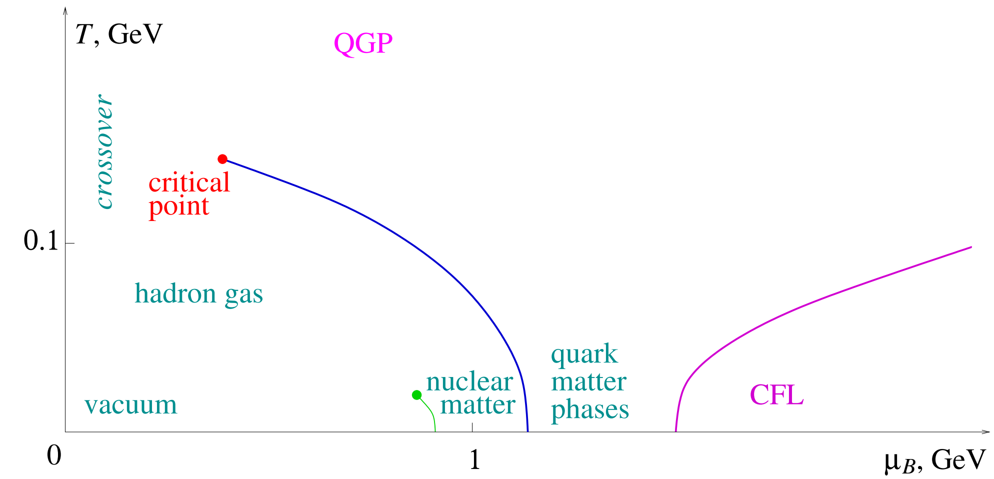
$$\downarrow$$

$$f_B, f_+^{D\to K}(q^2), B_K, \ldots$$

$$\downarrow$$

SUSY, Technicolor, LittleHiggs, ...

high temperature/density frontier



On either avenue, difficulties associated with strong interactions:

$$\mathcal{L}_{\mathrm{QCD}}\Big|_{\mathrm{mink}} = -\frac{1}{2}\mathrm{Tr}(F_{\mu\nu}F_{\mu\nu}) + \sum_{i=1}^{N_f} \bar{q}^{(i)}(\mathrm{i}\not{D} - m^{(i)})q^{(i)} \qquad \text{[FGL]}$$

How can one quantitatively master a theory with two different faces:

- fundamental degrees of freedom (quarks, gluons)
- collective/effective degrees of freedom (π, K, η, ...)

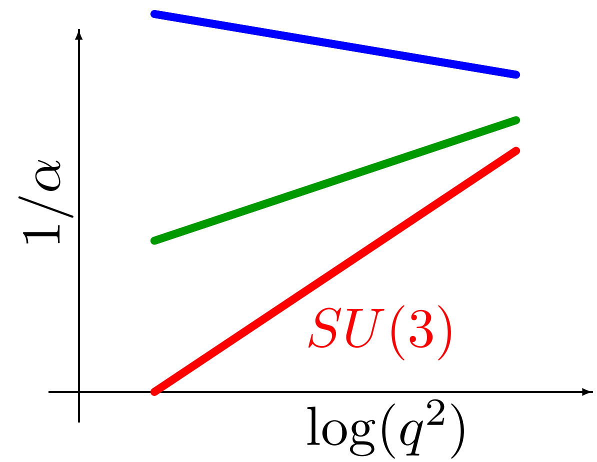
Why to combine LQCD and XPT

matter:

ν_e	ν_μ	ν_τ
e	μ	τ
u	c	t
d	s	b

forces:

$$\underbrace{U(1) \times SU(2)}_{\text{EW}} \times \underbrace{SU(3)}_{\text{QCD}}$$



The "two faces" of QCD are associated with

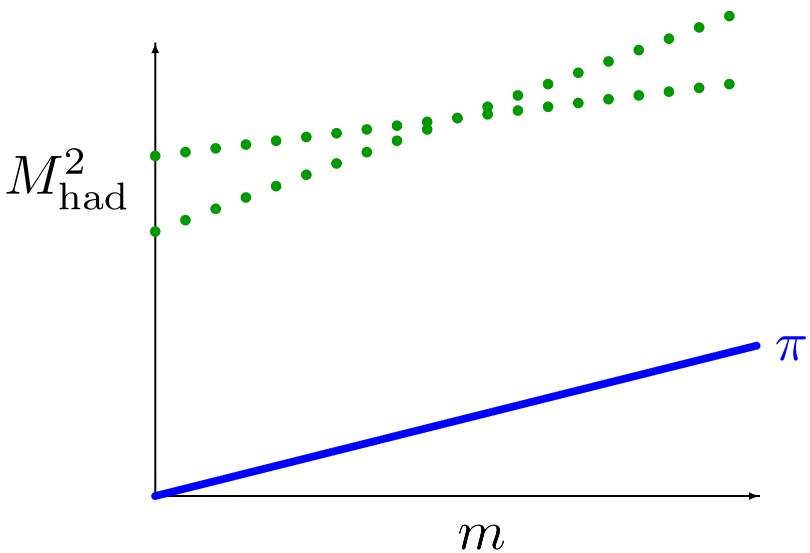
- asymptotic freedom at $q^2 \to \infty$ ("weak coupling regime" w.r.t. g^2)
- confinement & chiral symmetry breaking at $q^2 \to 0$ ("strong coupling regime")

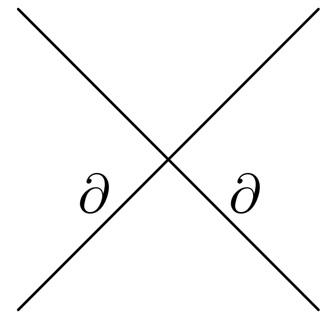
$$\mathcal{L}_{\text{QCD}}\Big|_{\text{eucl}} = \frac{1}{2}\text{Tr}(F_{\mu\nu}F_{\mu\nu}) + \sum_{i=1}^{N_f} \bar{q}^{(i)}(\not{D} + m^{(i)})q^{(i)}$$

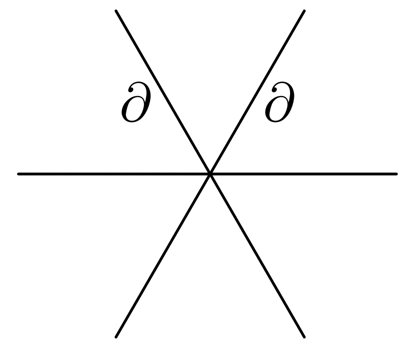
Use $\left\{ \begin{array}{c} \text{PQCD at high energy/momentum} \\ \text{XPT at low energy/momentum} \end{array} \right\}$ and LQCD throughout

Characteristic properties of QCD at low energy

Dominant degrees of freedom are “pions” (π, K, η, ...) with peculiar properties:







$\sim \partial\pi^a \partial\pi^b\, \pi^c \pi^d$ $\qquad$ $\sim \partial\pi\partial\pi\; \pi\pi\pi\pi$

$\Longrightarrow$ pion mass proportional to the square-root of quark mass

$$M_\pi^2 = B(m^{(1)} + m^{(2)}) \qquad \text{[GOR]}$$

$\Longrightarrow$ quarks interact softly at low energy – does this allow us to use weak-coupling techniques in strong-coupling regime of QCD ?

Symmetries of QCD Lagrangian

Consider $\mathcal{L}_{\mathrm{QCD}}$ in chiral limit $[m^{(i)} \to 0\ \forall i]$ and introduce $q^{(i)}_{R,L} = \frac{1}{2}(1 \pm \gamma_5) q^{(i)}$:

$$\sum_{i=1}^{N_f} \bar{q}^{(i)} \Big(\gamma_\mu \partial_\mu + i g \gamma_\mu A^a_\mu \frac{\lambda^a}{2} \Big) q^{(i)} = \sum_{i=1}^{N_f} \bar{q}^{(i)}_R (...) q_R + \bar{q}^{(i)}_L (...) q_L = q_R (...) q_R + q_L (...) q_L$$

This $\mathcal{L}_{m=0}$ with $q = (u, d, [s])^{\mathrm{T}}$ has the following global symmetries ($T^a = \frac{\sigma^a}{2}, \frac{\lambda^a}{2}, ...$):

$$\begin{aligned}
U(1)_V: \quad & q(x) \longrightarrow e^{\mathrm{i}\alpha} q(x) \\
& V^0_\mu \equiv \bar{q}\gamma_\mu q, \quad \partial_\mu V^0_\mu = 0 \longrightarrow \text{baryon number conservation} \\
U(1)_A: \quad & q(x) \longrightarrow e^{\mathrm{i}\beta\gamma_5} q(x) \\
& A^0_\mu \equiv \bar{q}\gamma_\mu\gamma_5 q, \quad \partial_\mu A^0_\mu = N_f \frac{g^2}{16\pi^2} \mathrm{Tr}(G_{\mu\nu}\tilde{G}_{\mu\nu})
\end{aligned}$$

$$\left.\begin{aligned}
SU(N_f)_V: \quad & q(x) \longrightarrow e^{\mathrm{i}\alpha T^a} q(x) \\
& V^a_\mu \equiv \bar{q}\gamma_\mu T^a q, \quad \partial_\mu V^a_\mu = 0 \\
SU(N_f)_A: \quad & q(x) \longrightarrow e^{\mathrm{i}\beta\gamma_5 T^a} q(x) \\
& A^a_\mu \equiv \bar{q}\gamma_\mu\gamma_5 T^a q, \quad \partial_\mu A^a_\mu = 0
\end{aligned}\right\} \text{chiral symmetry group } G$$

$\Longrightarrow$ gluons are flavor blind and do not change chirality of quark, quark mass does

Establishing the link: Linear sigma model

Toy model of SSB with 4 real fields $\vec{\phi} = (\phi^0, \phi^1, \phi^2, \phi^3)$, condensate v, quark mass h:

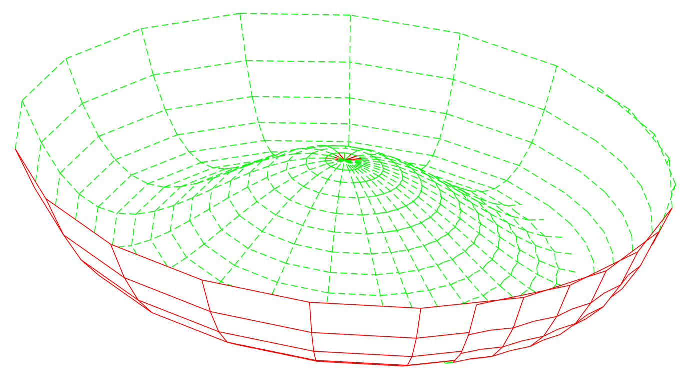
$$\mathcal{L} = \frac{1}{2}\partial_\mu\vec{\phi}\partial_\mu\vec{\phi} - \frac{g}{4}(\vec{\phi}^2 - v^2)^2 + h\phi_0$$

The $O(4)$ symmetry is broken spontaneously for $v^2 > 0$ and explicitly for $h > 0$. There are 3 ($h > 0$) or 6 ($h = 0$) conserved Noether currents

$$V_\mu^a = \epsilon^{abc}\phi^b\partial_\mu\phi^c\ , \quad A_\mu^a = \frac{1}{2}\partial_\mu\phi^0\phi^a - \frac{1}{2}\phi^0\partial_\mu\phi^a \qquad (a,b,c = 1,2,3)\ .$$

With $v^2 > 0, h > 0$ the minimum of the potential is at $\phi^0 = v + \frac{h}{2gv^2}$, $\vec{\phi} = \vec{0}$.
Introduce $\phi^0 = v + \frac{h}{2gv^2} + \sigma$, $(\phi^1, \phi^2, \phi^3) = \vec{\pi}$ and rewrite Lagrangian as

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\partial_\mu\sigma\partial_\mu\sigma - \frac{1}{2}[2gv^2 + \frac{3h}{v}]\sigma^2 \\
&+ \frac{1}{2}\partial_\mu\pi\partial_\mu\pi - \frac{h}{2v}\pi^2 \\
&- [gv + \frac{h}{2v^2}]\sigma(\sigma^2 + \pi^2) - \frac{g}{4}(\sigma^2 + \vec{\pi}^2)^2 - hv
\end{aligned}$$



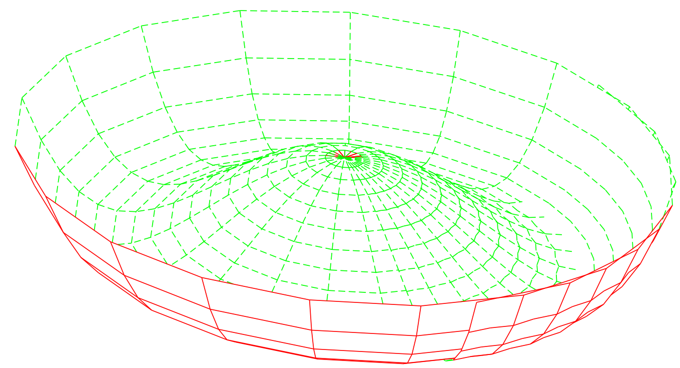
$\longrightarrow$ $M_\pi^2 = \frac{h}{v}$: linear in quark mass, hence $M_\pi^2 = 0$ at $h = 0$

$\longrightarrow$ $M_\sigma^2 = 2gv^2 + \frac{3h}{v}$: nonvanishing at $h = 0$, three-fold slope in h

$\longrightarrow$ $\pi^2\sigma$, $\pi^2\sigma^2$, π^4 vertices even for $h = 0$ is misleading

Introduce $\begin{pmatrix} \phi^0+\phi^3 & \phi^1+\mathrm{i}\phi^2 \\ \phi^1-\mathrm{i}\phi^2 & \phi^0-\phi^3 \end{pmatrix} = (v + \frac{h}{2gv^2} + \sigma)\, e^{\mathrm{i}\vec{\pi}\vec{\lambda}/(2v)}$ and rewrite Lagrangian as

$$\begin{aligned}
\mathcal{L} \;&=\; \frac{1}{2}\partial_\mu\sigma\partial_\mu\sigma - \frac{1}{2}[2gv^2 + \frac{3h}{v}]\sigma^2 \\
&+\; \frac{1}{2}\partial_\mu\pi\partial_\mu\pi - \frac{h}{2v}\pi^2 \\
&+\; \frac{1}{2v^2}(\sigma^2 + 2v\sigma)\partial_\mu\vec{\pi}\partial_\mu\vec{\pi} - \frac{g}{4}(\sigma^4 + 2v\sigma^3) + \ldots
\end{aligned}$$



$\longrightarrow$ M_π^2 and M_σ^2 as before

$\longrightarrow$ $\partial\pi\partial\pi\sigma^n$ vertices instead of $\pi^2\sigma^n$, i.e. derivative couplings

- In the presence of SSB, things are only simple if we choose adequate coordinates
- Pion field $U = \exp(\mathrm{i}\pi^a T^a/F)$ describes local fluctuation of condensate

Theoretical framework for SSB

Goal: understand Goldstone theorem with its implications for QCD

Noether theorem

Relation between symmetry and conservation law (version for global internal symm.):

$\mathcal{L} = \mathcal{L}(\phi_i, \partial_\mu \phi_i)$ invariant under symmetry group G:

finite trafo: $\phi_i(x) \longrightarrow \phi'_i(x) = (e^{\mathrm{i}\epsilon^a T^a})_{ij}\phi_j(x)$

infinitesimal: $\phi_i(x) \longrightarrow \phi'_i(x) = \phi_i(x) + \mathrm{i}\epsilon^a T^a_{ij}\phi_j(x)$

$[T^a, T^b] = \mathrm{i}f^{abc}T^c$ defines LA of G with structure constants f^{abc}

$\boxed{\partial_\mu J^a_\mu = 0 \text{ with } J^a_\mu = -\mathrm{i}\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_i)}T^a_{ij}\phi_j}$ Noether theorem (proof uses EOM)

$\boxed{\frac{d}{dt}Q^a = 0 \text{ with } Q^a(t) = \int_{t\,\mathrm{fixed}} J^a_0(\vec{x})\; d^3\vec{x}}$ Noether charge is conserved

$\boxed{[Q^a, Q^b] = \mathrm{i}f^{abc}Q^c}$ Noether charges satisfy LA of G (they are the generators)

Current algebra

Extension to case with additional (small) explicit symmetry breaking

$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ with $\mathcal{L}_0$ invariant under G and $\mathcal{L}_1$ not (e.g. quark mass term in QCD)
$Q^a(t) \equiv \int J_0^a(t,\vec{x})\; d^3\vec{x} = -\mathrm{i} \int \frac{\delta L}{\delta(\partial_0 \phi_i)} T^a_{ij} \phi_j \; d^3\vec{x}$

$$\boxed{[Q^a(t), Q^b(t)] = \mathrm{i} f^{abc} Q^c(t)} \qquad \text{Charge/current algebra}$$

In the presence of symmetry breaking terms, similar equal-time commutation relations hold between t-dependent charges; they still serve as generators of the broken symm.!

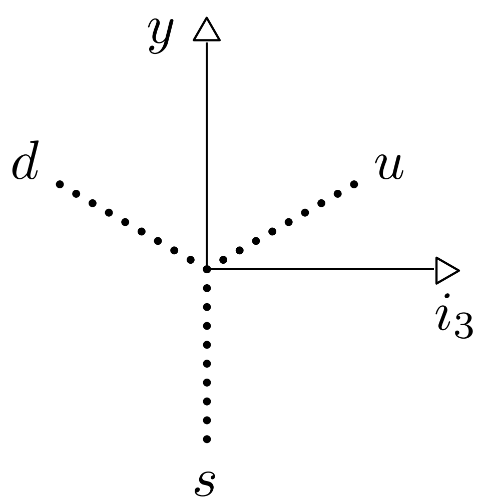
Symmetry fates/realizations

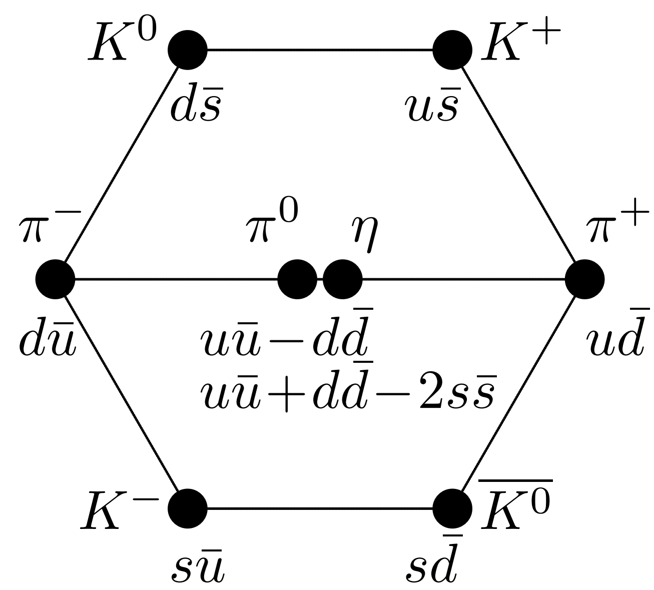
Let Q be a conserved charge: $[Q, P_\mu] = 0, J_0(\vec{x}) = e^{\mathrm{i}\vec{P}\vec{x}} J_0(0) e^{-\mathrm{i}\vec{P}\vec{x}}$. Then:

$$\begin{aligned} ||Q|0\rangle||^2 &= \langle 0|QQ|0\rangle = \int \langle 0|Q J_0(\vec{x})|0\rangle \; d^3\vec{x} \\ &= \int \langle 0|Q e^{\mathrm{i}\vec{P}\vec{x}} J_0(\vec{0}) e^{-\mathrm{i}\vec{P}\vec{x}}|0\rangle \; d^3\vec{x} = \int \langle 0|Q J_0(\vec{0})|0\rangle \; d^3\vec{x} = \left\{ {0 \atop \infty} \right. \end{aligned}$$

Wigner-Weyl (WW): $Q\lvert 0\rangle = 0$	Nambu-Goldstone (NG): $Q\lvert 0\rangle \neq 0$
explicitly realized symmetry	spontaneously broken symmetry
linear representation	non-linear representation
degenerate multiplets	massless Goldstone bosons appear

Educated guess for QCD





$$
\begin{aligned}
[Q_L^a(t), Q_L^b(t)] &= \mathrm{i} f^{abc} Q_L^c(t) \\
[Q_R^a(t), Q_R^b(t)] &= \mathrm{i} f^{abc} Q_R^c(t) \\
[Q_L^a(t), Q_R^b(t)] &= 0
\end{aligned}
\qquad
\begin{aligned}
[Q_V^a(t), Q_V^b(t)] &= \mathrm{i} f^{abc} Q_V^c(t) \\
[Q_A^a(t), Q_A^b(t)] &= \mathrm{i} f^{abc} Q_V^c(t) \\
[Q_V^a(t), Q_A^b(t)] &= \mathrm{i} f^{abc} Q_A^c(t)
\end{aligned}
$$

$$
\underbrace{SU(N_f)_L \times SU(N_f)_R}_{\text{chiral group } G} = \underbrace{SU(N_f)_V}_{\text{subgroup } H} \times \underbrace{SU(N_f)_A}_{\text{coset } G/H}
$$

Educated guess: $\underbrace{H \text{ is realized à la WW}}_{\text{eightfold way / } SU(N_f) \text{ pattern}}$ $\quad$ $\underbrace{G/H \text{ is realized à la NG}}_{\pi,K,\eta \text{ light / } N_f^2-1 \text{ Goldstone bosons}}$

Goldstone theorem

$$
\begin{aligned}
G: \quad \phi_i(x) \longrightarrow \phi'_i(x) &= e^{-\mathrm{i}\epsilon^a Q^a}\phi_i(x)e^{\mathrm{i}\epsilon^a Q^a} \\
&= (U(g)\phi)_i = U(g)_{ij}\phi_j = (e^{\mathrm{i}\epsilon^a T^a})_{ij}\phi_j(x)
\end{aligned}
$$

with $[Q^a, Q^b] = \mathrm{i}f^{abc}Q^c$ and $[T^a, T^b] = \mathrm{i}f^{abc}T^c$

$e^{\mathrm{i}\epsilon Q^k}|0\rangle = |0\rangle$ for $k \in \{1, ..., \dim(H)\}$ [WW, $\rightarrow$ 8 V-charges]

$e^{\mathrm{i}\epsilon Q^\ell}|0\rangle = |0\rangle$ for $\ell \in \{\dim(H)+1, ..., \dim(G)\}$ [NG, $\rightarrow$ 8 A-charges]

- Let ϕ be a field with $\lim_{V\to\infty}\langle 0|\,[Q^\ell, \phi]\,|0\rangle \neq 0$
- There is a state $|\pi\rangle$ with $\langle 0|J_0^\ell(0)|\pi\rangle\langle\pi|\phi|0\rangle \neq 0$ and $P^2|\pi\rangle = 0$
- Theory has $\dim(G/H) = \dim(G) - \dim(H)$ massless Goldstone bosons
  [QCD: Q^ℓ any of the 8 generators of $SU(3)_A$ yields 8 "pions"]
  [$O(N) \to O(N-1)$ yields $\frac{N(N-1)}{2} - \frac{(N-1)(N-2)}{2} = N - 1$ "pions"]

Order parameters of SSB

Goldstone theorem: NG realization $\longleftrightarrow \exists\phi_j$ with $\langle 0|\phi_j|0\rangle \neq 0$

$$[Q^a,\phi_i] = T^a{}_{ij}\phi_j \qquad \left\{ \begin{array}{lll} \phi_i \longrightarrow e^{-\mathrm{i}\epsilon^a Q^a}\phi_i(x)e^{\mathrm{i}\epsilon^a Q^a} & = \phi_i - \mathrm{i}\epsilon^a[Q^a,\phi_i] + \ldots \\ \phi_i \longrightarrow (e^{-\mathrm{i}\epsilon^a T^a})_{ij}\phi_j(x) & = \phi_i - \mathrm{i}\epsilon^a T^a{}_{ij}\phi_j \; + \ldots \end{array} \right.$$

$$\underbrace{\langle 0|\,[Q^a,\phi_i]\,|0\rangle}_{=0\,,\,\neq 0} = \underbrace{T^a{}_{ij}\langle 0|\phi_j|0\rangle}_{=0\,,\,\neq 0}$$

The difference $=0$ versus $\neq 0$ corresponds to the cases $a \rightarrow k$ (explicit symmetry) versus $a \rightarrow \ell$ (hidden symmetry)

$$\boxed{\text{Order parameter } \langle 0|\phi_j|0\rangle \text{ transforms } \left\{ \begin{array}{c} \text{trivially under } H \\ \text{non-trivially under } G/H \end{array} \right.}$$

QCD: d=3: $\langle 0|\bar{q}q|0\rangle$ qualifies, $\langle 0|\bar{q}\gamma_5 q|0\rangle$ does not

d=4: $\langle 0|\mathrm{Tr}(G_{\mu\nu}G_{\mu\nu})|0\rangle$ not qualifying for $SU(3)_L \times SU(3)_R \longrightarrow SU(3)_V$

d=5: $\langle 0|\bar{q}\sigma_{\mu\nu}G_{\mu\nu}q|0\rangle$ OK, $\langle 0|\bar{q}q\bar{q}q|0\rangle$ OK, ...

Pion pole dominance and soft-coupling theorems

$$\text{Goldstone: } \langle 0|A^a_\mu(0)|\pi^b(\vec{p})\rangle = \mathrm{i}F^{ab}_\pi p_\mu = \mathrm{i}\delta^{ab}F_\pi p_\mu$$

$$\text{derivative: } \langle 0|\partial_\mu A^a_\mu(0)|\pi^b(\vec{p})\rangle = \underbrace{\delta^{ab}}_{\langle 0|\phi^a|\pi^b\rangle} F_\pi \underbrace{p^2}_{M_\pi^2}$$

PCAC: $\boxed{\partial_\mu A^a_\mu = F_\pi M_\pi^2 \phi^a}$ $\left\{ \begin{array}{l} \text{hypothesis concerns off-shell behavior} \\ \partial_\mu A^a_\mu \text{ sensitive to expl. symm. breaking} \end{array} \right.$

- Goldstone bosons generate singularities in Green functions; e.g. poles at $p^2=0$ from one-pion-states: $\int \langle 0|T\{A^a_\mu(x)A^a_\mu(0)|0\rangle\} e^{\mathrm{i}px}\,dx = \delta^{ab}(p_\mu p_\nu - g_{\mu\nu}p^2)\Pi_{AA}(p^2)$ with $\Pi_{AA}(p^2) = F_\pi^2/(p^2+\mathrm{i}0) + \mathrm{const} + O(p^2)$.

- Lorentz covariance plus CVC/PCAC imply soft coupling theorems:

$$p^\mu\langle \pi^{a_1}(\vec{p}_1)\,...\,\pi^{a_{2n}}(\vec{p}_{2n})|V^a_\mu|0\rangle = 0 \qquad [\partial_\mu V^a_\mu = 0,\ p_\mu = (p_1+...+p_{2n})_\mu]$$

$$p^\mu\langle \pi^{a_1}(\vec{p}_1)\,...\,\pi^{a_{2n+1}}(\vec{p}_{2n+1})|A^a_\mu|0\rangle = 0 \qquad [\partial_\mu A^a_\mu = 0,\ p_\mu = (p_1+...+p_{2n+1})_\mu]$$

$$n=2:\ \langle \pi^{a_1}(\vec{p}_1)\pi^{a_2}(\vec{p}_2)|V^a_\mu(0)|0\rangle = \tfrac{\mathrm{i}C_\pi p_\mu}{p^2+\mathrm{i}0}\,g_{a_1a_2a}(p_1,p_2) + ...$$

$$p_\mu\langle ...|V^a_\mu(0)|0\rangle = -\mathrm{i}C_\pi g_{a_1a_2a}(p_1,p_2) + ... \overset{p_1,p_2\to 0}{=} 0 \longrightarrow g_{a_1a_2a}(0,0)=0$$

$$n=3:\ \langle \pi^{a_1}(\vec{p}_1)...\pi^{a_3}(\vec{p}_3)|A^a_\mu(0)|0\rangle = \tfrac{\mathrm{i}D_\pi p_\mu}{p^2+\mathrm{i}0}\,g_{a_1a_2a_3a}(p_1,p_2,p_3) + ...$$

$$p_\mu\langle ...|A^a_\mu(0)|0\rangle = -\mathrm{i}D_\pi g_{a_1a_2a_3a}(p_1...p_3) + ... \overset{p_{1..3}\to 0}{=} 0 \to g_{a_1a_2a_3a}(0,0,0)=0$$

Effective Lagrangian construction

Pion field $U(x)$ parametrizes Goldstone manifold G/H; trafo law from group theory

$$U(x) = e^{\mathrm{i}\sqrt{2}\phi(x)/F}$$
$$F = 86.2 \pm 0.5\,\mathrm{MeV}$$

$$\phi(x) = \phi^a T^a = \phi^a \frac{\lambda^a}{2} = \begin{pmatrix} \frac{\eta}{\sqrt{6}} + \frac{\pi^0}{\sqrt{2}} & \pi^+ & K^+ \\ \pi^- & \frac{\eta}{\sqrt{6}} - \frac{\pi^0}{\sqrt{2}} & K^0 \\ K^- & \overline{K^0} & -\frac{2\eta}{\sqrt{6}} \end{pmatrix}$$

$$\varphi : G \times G/H \longrightarrow G/H,\ (g,[n]) \longrightarrow ?,\ [n] = \{nh|h \in H\}$$

Decomposition $g = n_g h_g$ is unique; action on class representative n is $gn = n_g h_g n = n'h'$ and gives new n' in a unique manner

QCD: $G = SU(3)_L \times SU(3) + R$, $H = SU(3)_V$, $G/H = SU(3)_A$

$$g \equiv (V_R, V_L),\ h \equiv (V,V),\ n \equiv (U,1),\ n' \equiv (U',1)$$

$$\longrightarrow\ gn = (V_R,V_L)(U,1) = (V_R U, V_L) = (V_R U V_L^\dagger, 1)(V_L, V_L) \equiv n'h'$$

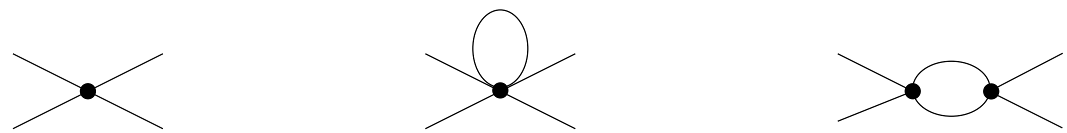
$$\boxed{U \to U' = V_R U V_L^\dagger}$$

π transforms non-linearly
U transforms bi-linearly

Chiral Lagrangian $O(p^2)$

Goal: construct the most general Lagrangian built from $U(x)$ and $U(x)^\dagger$ which is consistent with all symmetries of the underlying QCD Lagrangian.

Expansion in external momentum $p^2, p^4, \ldots$; start with tree-level processes; unpacking $U \to \pi$ needed



(a) (b) (c)

Use $\langle \ldots \rangle$ to denote trace in flavor space and $M = \mathrm{diag}(m_u, m_d, m_s)$ with $m_{u,d,s} > 0$

$$\mathcal{L}^{(2)} = \frac{F^2}{4} \langle\, \partial_\mu U \partial_\mu U^\dagger + 2BM\,U + 2BM\,U^\dagger \,\rangle$$

- massless part invariant under global $U \to V_R U$ and $U \to U V_L^\dagger$
- massive part invariant under global $U \to V U V^\dagger$
- expansion yields $\mathcal{L}^{(2)} = \frac{1}{2}\partial_\mu \pi^0 \partial_\mu \pi^0 + \partial_\mu \pi^+ \partial_\mu \pi^- - Bm\pi^0\pi^0 - 2Bm\pi^+\pi^- + \ldots$
- tree-level π-π-scattering vertex is $\frac{1}{6F^2}[\pi^a \partial_\mu \pi^a \pi^b \partial_\mu \pi^b - \pi^a \pi^a \partial_\mu \pi^b \partial_\mu \pi^b]$
- just 2 low-energy constants ($F = \lim_{m\to 0} F_\pi$, $B = \Sigma/F$) determine all interactions
- spurion formalism brings $\partial_\mu \to \nabla_\mu$ and $2BM \to \chi$ and $F^R_{\mu\nu}, F^L_{\mu\nu}$ [external currents]

Light quark mass ratios

$\mathcal{L}^{(2)}$ and higher order Lagrangians contain only products $Bm_{u,d,s}$, which are scheme- and scale-invariant quantities [RGI]; with $m_{ud} = (m_u + m_d)/2$ [isospin limit]:

$$\begin{aligned} M_{\pi^\pm}^2 &= B2m_{ud} \quad , \qquad M_{\pi^0}^2 = B2m_{ud} + O\Big(\frac{[m_u - m_d]^2}{m_s - m_{ud}}\Big) \\ M_{K^\pm} &= B(m_u + m_s) \quad , \qquad M_{K^0} = B(m_d + m_s) \\ M_\eta^2 &= B(\frac{2}{3}m_{ud} + \frac{4}{3}m_s) + O\Big(\frac{[m_u - m_d]^2}{m_s - m_{ud}}\Big) \end{aligned}$$

This implies several well-known relations:

- $M_\pi^2 F_\pi^2 = \Sigma 2m_{ud}$ with $\Sigma = -\langle \bar{u}u \rangle = -\langle \bar{d}d \rangle = -\langle \bar{s}s \rangle$ $(\lim_{m\to 0})$ [GOR]
- $B = \frac{M_\pi^2}{2m_{ud}} = \frac{M_K^2}{m_s+m_u} = \frac{M_\eta^2}{m_s+m_d}$ [Weinberg]
- $3M_\eta^2 = 4M_K^2 - M_\pi^2$ [GellMann Okubo]

Quark mass ratios, as determined from phenomenology:

	m_u/m_d	m_s/m_d	m_s/m_{ud}
$O(p^2)$	0.55	20.1	25.9
$O(p^4)$	0.55 ± 0.04	18.9 ± 0.8	24.4 ± 1.5

Chiral Lagrangian $O(p^4)$

Ordering principle is $\partial^2 \sim m$ [or $\sim\chi$], as suggested by GOR. Recipe: construct, at any order, the most general Lagrangian which is consistent with all underlying symmetries.

$$\begin{aligned}
\mathcal{L}^{(4)} &= L_1\langle\nabla_\mu U\nabla_\mu U^\dagger\rangle^2 \\
&+ L_2\langle\nabla_\mu U\nabla_\nu U^\dagger\rangle\langle\nabla_\mu U\nabla_\nu U^\dagger\rangle \\
&+ L_3\langle\nabla_\mu U\nabla_\mu U^\dagger\rangle\langle\nabla_\nu U\nabla_\nu U^\dagger\rangle \\
&+ L_4\langle\nabla_\mu U\nabla_\mu U^\dagger\rangle\langle\chi^\dagger U+\chi U^\dagger\rangle \\
&+ L_5\langle\nabla_\mu U\nabla_\mu U^\dagger(\chi^\dagger U+\chi U^\dagger)\rangle \\
&+ L_6\langle\chi^\dagger U+\chi U^\dagger\rangle^2 + L_7\langle\chi^\dagger U-\chi U^\dagger\rangle^2 \\
&+ L_8\langle\chi^\dagger U\chi^\dagger U+\chi U^\dagger\chi U^\dagger\rangle \\
&- \mathrm{i}L_9\langle F^L_{\mu\nu}\nabla_\mu U\nabla_\nu U^\dagger + F^R_{\mu\nu}\nabla_\mu U\nabla_\nu U^\dagger\rangle \\
&+ L_{10}\langle UF^L_{\mu\nu}U^\dagger F^R_{\mu\nu}\rangle + H_1 ... + H_2 ...
\end{aligned}$$

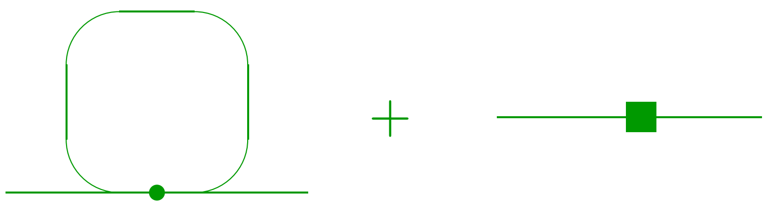
Do fully fledged QFT calculations with this Lagrangian [Weinberg, Gasser, Leutwyler]. L_i (for $N_f=3$) or ℓ_i (for $N_f=2$): Gasser-Leutwyler low-energy constants, undergo renomalization, their finite ($\propto\varepsilon^0$) pieces must be determined from experiment/lattice.

Dimensional regularization: $d = 4 - 2\varepsilon$

(1) Unpack U to generate Feynman rules in terms of π, K, η
(2) Draw all diagrams, to given external sources, that show up at this order
(3) Evaluate loop integrals in dimensional regularization

XPT: $\mathcal{L} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \ldots$ expansion in $p^2 \sim m$

$\Sigma_\pi^{\mathrm{NLO}} =$



Contributions to pion self-energy at NLO: 1-loop graph with a vertex from $\mathcal{L}^{(2)}$ [tiny dot] and a counterterm from $\mathcal{L}^{(4)}$ [fat box]. The divergent parts ($\propto \varepsilon^{-1}$) compensate each other, and in the finite parts ($\propto \varepsilon^0$) the μ-dependence cancels exactly.

$$L_i(d) = \frac{(c\mu)^{-2\varepsilon}}{(4\pi)^2}\Big\{ -\frac{\Gamma_i}{2\varepsilon} + L_i^{\mathrm{r[en]}}(\mu,[c,\varepsilon])\Big\}\,, \qquad \mu\frac{dL_i^{\mathrm{r}}(\mu)}{d\mu} = -\frac{\Gamma_i}{(4\pi)^2}$$

$\Gamma_1 = \frac{3}{32}$, $\Gamma_2 = \frac{3}{16}$, $\Gamma_3 = 0$, $\Gamma_4 = \frac{1}{8}$, $\Gamma_5 = \frac{3}{8}$, $\Gamma_6 = \frac{11}{144}$, $\Gamma_7 = 0$, $\Gamma_8 = \frac{5}{48}$, $\Gamma_9 = \frac{1}{4}$, $\Gamma_{10} = -\frac{1}{4}$

- Dimensional regularization respects all symmetries it should.
- One must(!) evaluate loop-integrals all the way up to infinity.
- For $N_f = 2$ often $\bar{\ell}_i = \frac{32\pi^2}{\gamma_i}\ell_i^{\mathrm{r}}(\mu) - \log\frac{M_\pi^2}{\mu^2}$ is used, depends on M_π instead of μ.

Order-by-order renormalizability

Example: VV-correlator to two loops

$O(p^2)$ contribution [vanishes for VV]

$O(p^4)$ contribution from (a-c):

1-loop diagrams with vertices from $\mathcal{L}^{(2)}$

0-loop diagrams with 1 vertex from $\mathcal{L}^{(4)}$

$O(p^6)$ contribution from (d-o):

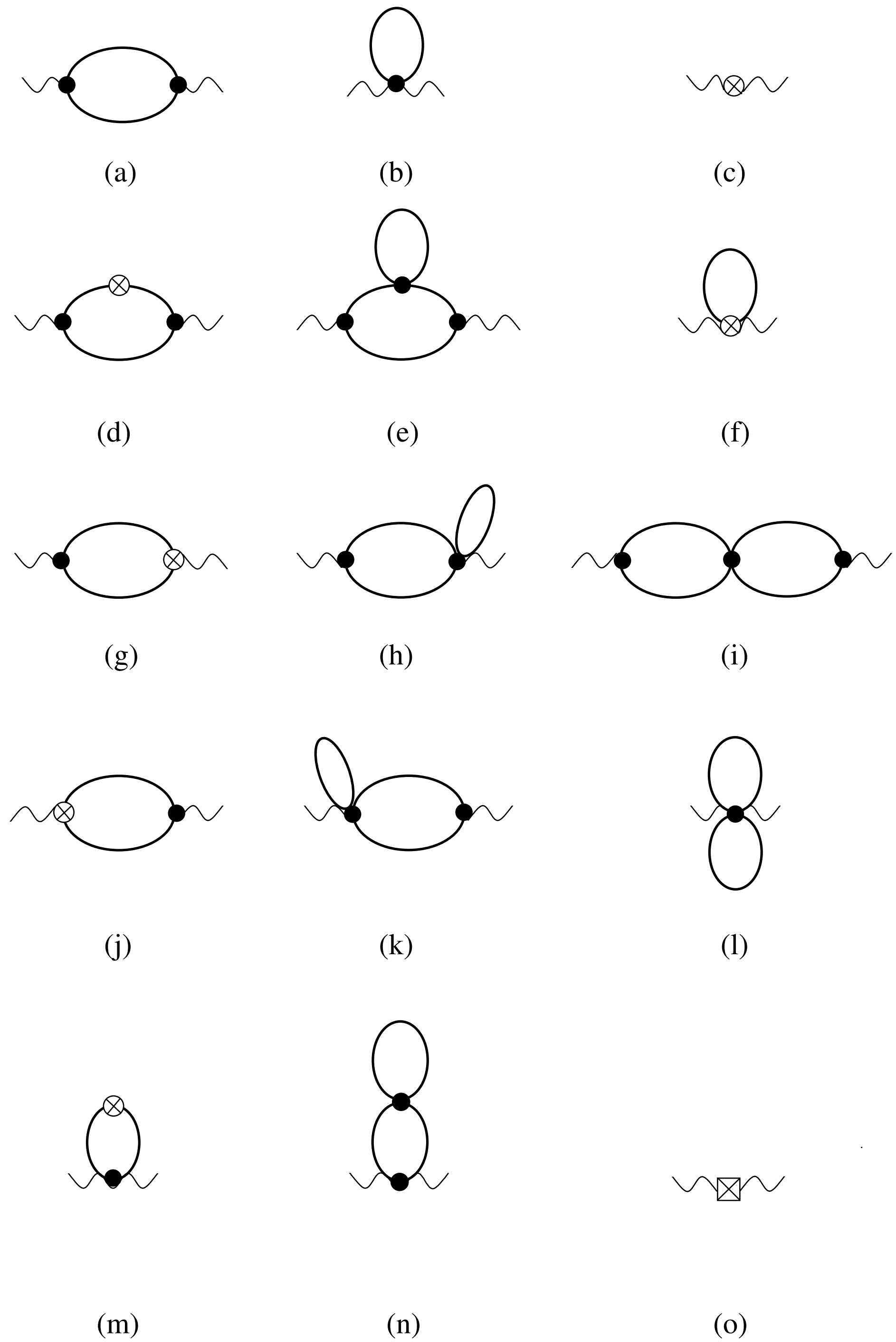
2-loop diagrams with vertices from $\mathcal{L}^{(2)}$

1-loop diagrams with 1 vertex from $\mathcal{L}^{(4)}$

0-loop diagrams with 2 vertices from $\mathcal{L}^{(4)}$

0-loop diagrams with 1 vertex from $\mathcal{L}^{(6)}$

When shrinking a given loop into a blob, there is a vertex from $\mathcal{L}^{(4)}$ or $\mathcal{L}^{(6)}$ to absorb its divergence: pieces $\propto \varepsilon^{-2}, \varepsilon^{-1}$ cancel, μ-dependence in $\propto \varepsilon^0$ drops out



Lattice application: chiral extrapolation

$$
\begin{aligned}
M_P^2 &= M^2\Big(1+\frac{1}{2}\frac{M^2}{(4\pi F)^2}\Big[\log(\frac{M^2}{\mu^2})+64\pi^2\ell_3^{\rm r}(\mu)\Big]\Big) \\
F_P &= F\ \Big(1-\ \ \frac{M^2}{(4\pi F)^2}\Big[\log(\frac{M^2}{\mu^2})-16\pi^2\ell_4^{\rm r}(\mu)\Big]\Big)
\end{aligned}
$$

$$
\begin{aligned}
M_P^2 &= M^2\Big(1+\frac{1}{2}\frac{M^2}{(4\pi F)^2}\Big[\log(\frac{M^2}{M_\pi^2})-\bar{\ell}_3\Big]\Big) \\
F_P &= F\ \Big(1-\ \ \frac{M^2}{(4\pi F)^2}\Big[\log(\frac{M^2}{M_\pi^2})-\bar{\ell}_4\Big]\Big)
\end{aligned}
$$

$$
\begin{aligned}
M_P^2 &= M^2\Big(1+\frac{1}{2}\frac{M^2}{(4\pi F)^2}\log(\frac{M^2}{\Lambda_3^2})\Big) \\
F_P &= F\ \Big(1-\ \ \frac{M^2}{(4\pi F)^2}\log(\frac{M^2}{\Lambda_4^2})\Big)
\end{aligned}
$$

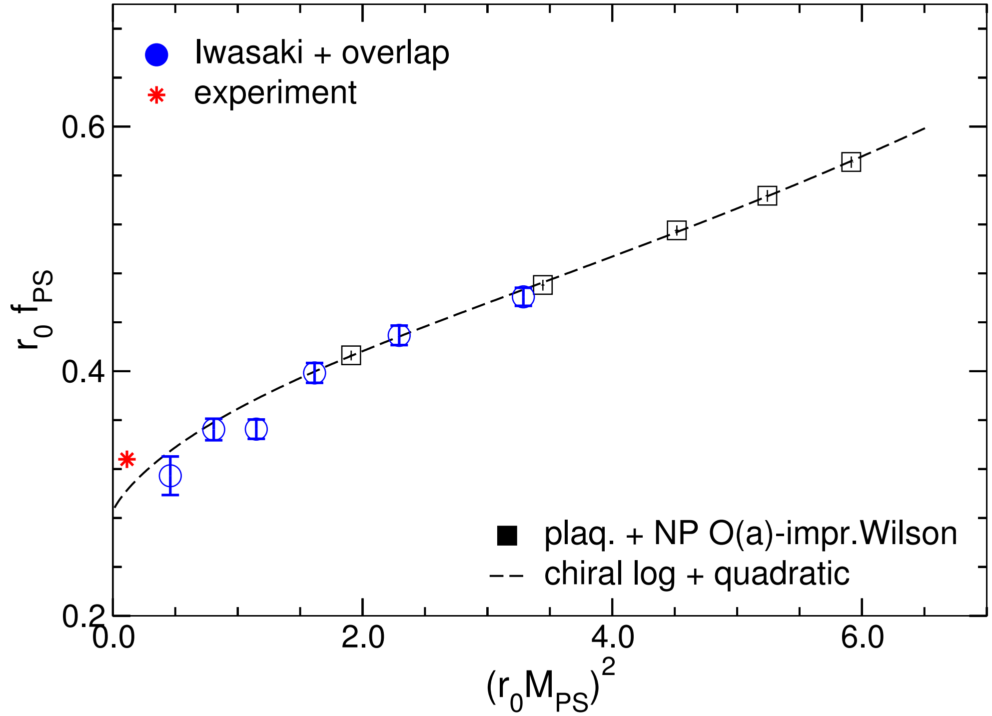
Last version practical: result depends on scale-independent quantities $F, M^2{=}2Bm, \Lambda_i$

$$M_\pi^2/(2Bm) = 1 + \frac{Bm}{(4\pi F)^2}\log(\frac{2Bm}{\Lambda_3^2})$$
$$F_\pi/F = 1 - \frac{2Bm}{(4\pi F)^2}\log(\frac{2Bm}{\Lambda_4^2})$$

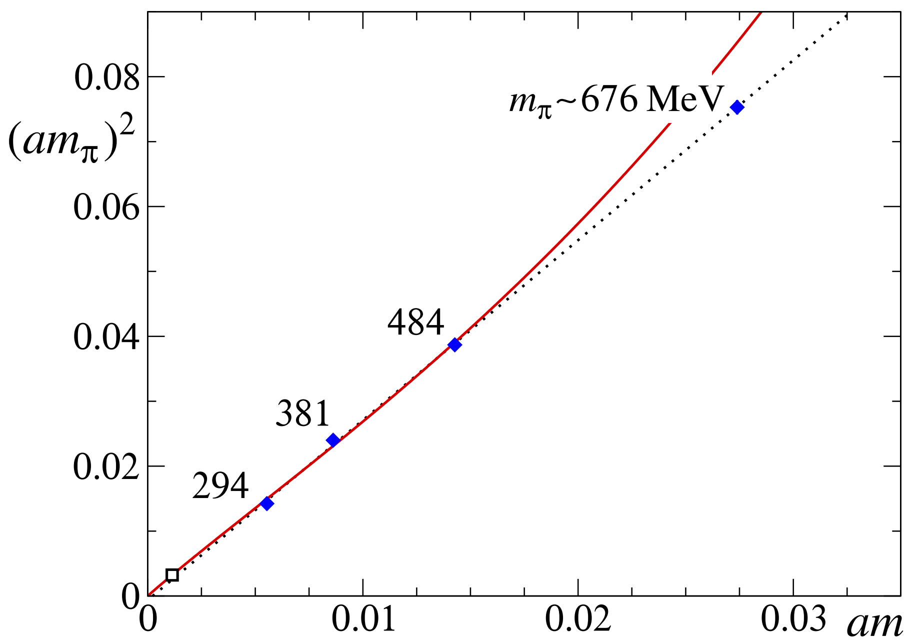
Chiral logs seem gently "switched off" at larger quark mass $\longrightarrow$ unstable fits.

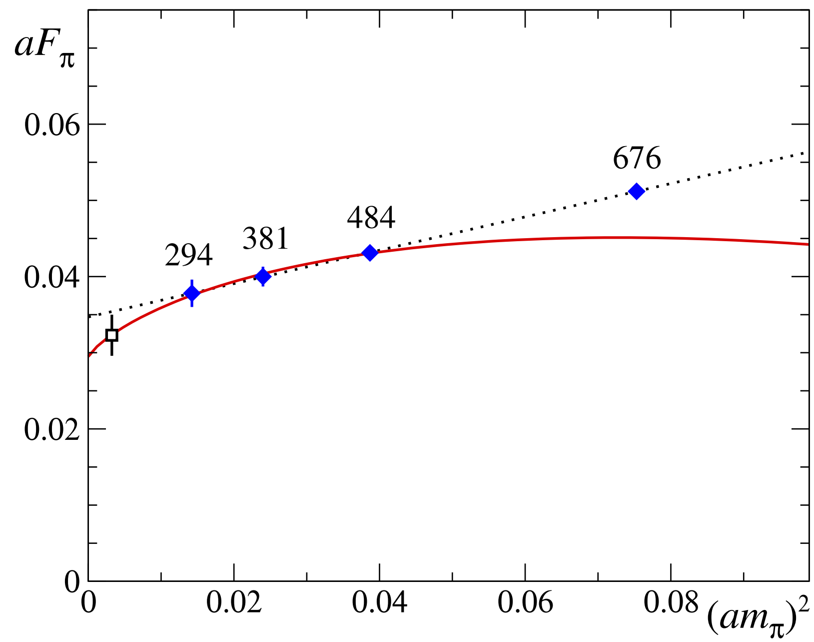
Finite-volume effects mimic chiral logs.

For lattice people "chiral extrapolation" means $m_{ud} \to m_{ud}^{\rm phys}$, not to 0.



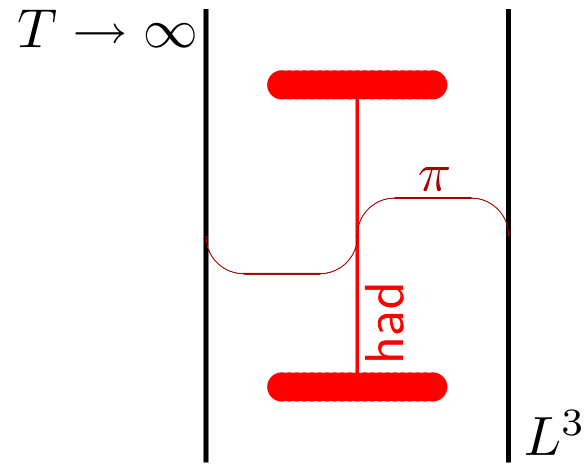
JLQCD, PoS (LAT2006) 054 [hep-lat/0610036]





M. Lüscher, PoS (LAT2005) 002

Lattice application: finite volume effects



idea: $$M_{\rm had}(\infty) = \underbrace{M_{\rm had}(L)}_{\text{lattice}} \cdot \underbrace{\frac{M_{\rm had}(\infty)}{M_{\rm had}(L)}}_{\text{EFT}}$$

core: $$\frac{M_{\rm had}(\infty)}{M_{\rm had}(L)} = 1 - \text{const}\; e^{-M_\pi L} \quad (\forall\, \text{had})$$

In finite (spatial) volume $V = L^3$ only momenta $\vec{p} = \frac{2\pi}{L}\vec{n},\ \vec{n} \in \mathbf{Z}^3$ possible

Two <u>basic conditions</u> for XPT in finite volume: $(\Lambda_{\rm XPT} \simeq 4\pi F_\pi \simeq 1\,\text{GeV})$

(1) $m \ll \Lambda_{\rm XPT}$ or $M_\pi \ll 4\pi F_\pi$ (2) $\frac{2\pi}{L} \ll \Lambda_{\rm XPT}$ or $1 \ll 2F_\pi L$

Once satisfied, still <u>two varieties</u> for pion correlation length:

(3a) $M_\pi L \gg 1$: $M_\pi^2 \sim \frac{1}{L^2} \sim m$ "p-regime" for $T \to \infty$

(3b) $M_\pi L \leq 1$: $M_\pi^2 \sim \frac{1}{L^4} \sim m$ "ϵ-regime" / "δ-regime"

Finite-volume shifts in $M_\pi(L)$, $F_\pi(L)$

$\Sigma_\pi^{\mathrm{NLO}}(L) - \Sigma_\pi^{\mathrm{NLO}}(\infty) \quad =$

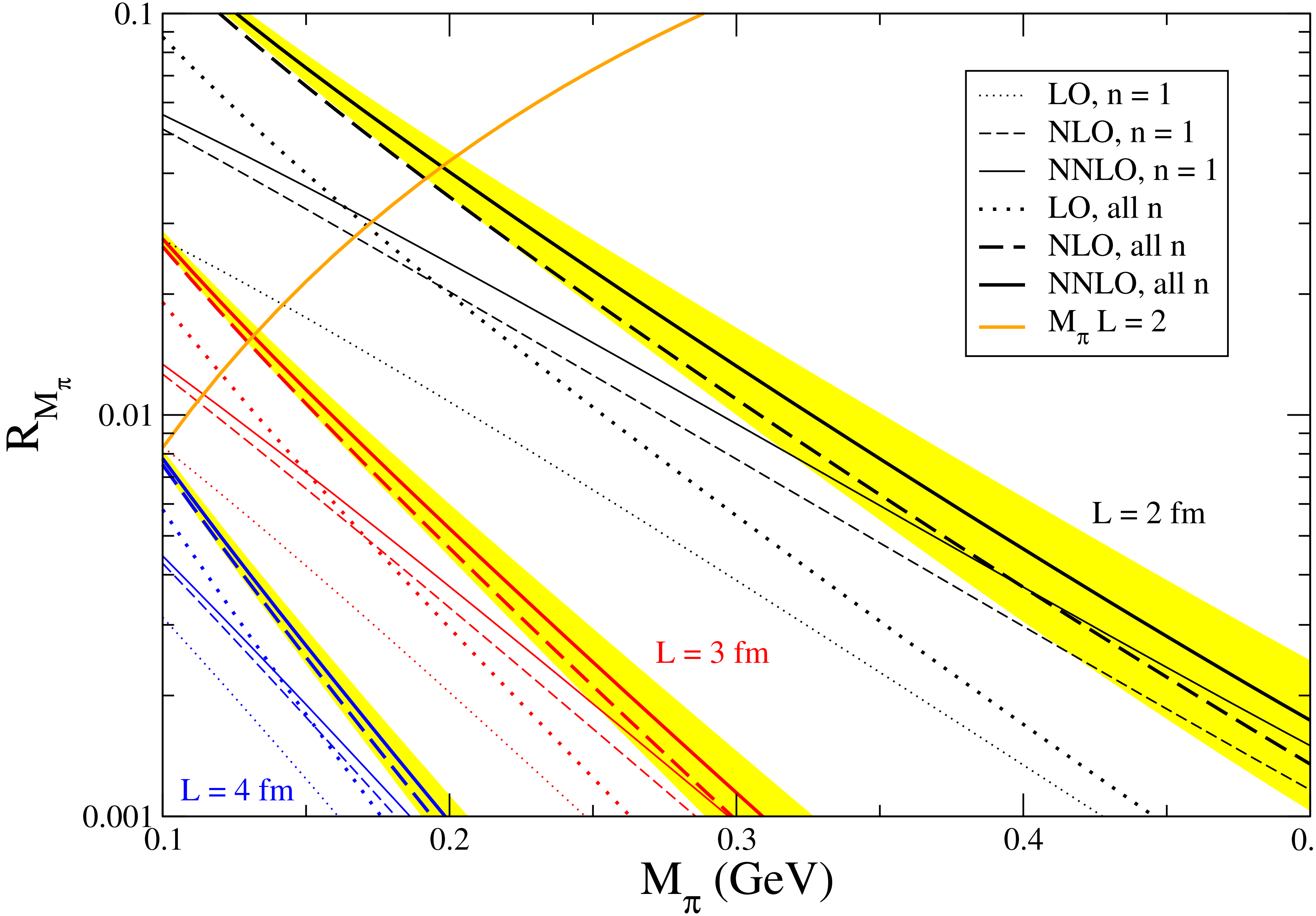
$\mathcal{L}^{\mathrm{NLO}}$ counterterm drops out and $G_L(x) - G_\infty(x) = \sum_{\vec{n}\in\mathbf{Z}^3\backslash\vec{0}} G_\infty(x^0, \vec{x}+\vec{n}L)$ remains

$$
\begin{aligned}
M_\pi(L) &= M_\pi \left[1 + \frac{1}{2N_f}\xi\tilde{g}_1(\lambda) + O(\xi^2)\right] \\
F_\pi(L) &= F_\pi \left[1 - \frac{2}{N_f}\xi\tilde{g}_1(\lambda) + O(\xi^2)\right]
\end{aligned}
$$

where $N_f \geq 2$, $M_\pi = M_\pi(\infty)$, $F_\pi = F_\pi(\infty)$, $\xi = \frac{M_\pi^2}{(4\pi F_\pi)^2}$, $\lambda = M_\pi L$ and

$$
\tilde{g}_1(\lambda) = \int_0^\infty \sum_{\vec{n}\in\mathbf{Z}^3\backslash\vec{0}} e^{-\frac{1}{\alpha}-\frac{\alpha}{4}\vec{n}^2\lambda^2}\, d\alpha = \sum_{n=1}^{\infty} \frac{4m(n)}{\sqrt{n}\lambda} K_1(\sqrt{n}\lambda)
$$

with $m(n)$ the multiplicity of vectors with $|\vec{n}^2| = n$



Memo: $M_\pi(L)/M_\pi = R_{M_\pi} + 1$

Lattice extension: partially quenched framework

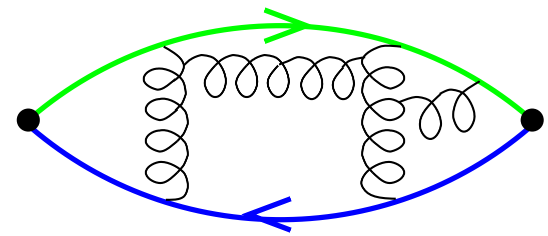
Hadronic correlator in $N_f \geq 2$ QCD: $C(t) = \int d^4x\, C(t,\mathbf{x})\, e^{\mathrm{i}\mathbf{px}}$ with

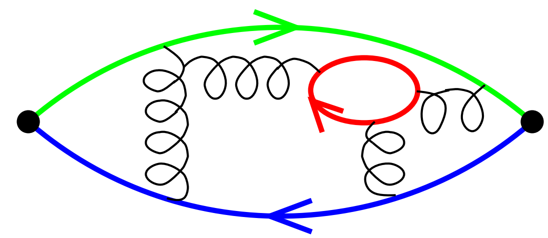
$$C(x) = \langle O(x)\, O(0)^\dagger \rangle \;=\; \frac{1}{Z}\int DU\, D\bar{q}Dq\;\; O(x)\, O(0)^\dagger\;\; e^{-S_G-S_F}$$

where $O(x) = \bar{d}(x)\Gamma u(x)$ and $\Gamma = \gamma_5, \gamma_4\gamma_5$ for $\pi^\pm$ and
$S_G = \beta \sum (1 - \frac{1}{3}\mathrm{Re}\mathrm{Tr}\, U_{\mu\nu}(x))$, $S_F = \sum \bar{q}(D+m)q$

$$\langle \bar{d}(x)\Gamma_1 u(x)\;\bar{u}(0)\Gamma_2 d(0)\rangle = \frac{1}{Z}\int DU\;\; \det(D+m)^{N_f}\; e^{-S_G}$$

$$\times \mathrm{Tr}\Big\{\Gamma_1 (D+m)^{-1}_{x0}\Gamma_2\; \underbrace{(D+m)^{-1}_{0x}}_{\gamma_5[(D+m)^{-1}_{x0}]^\dagger\gamma_5}\Big\}$$


(A) Quenched QCD: quark loops neglected

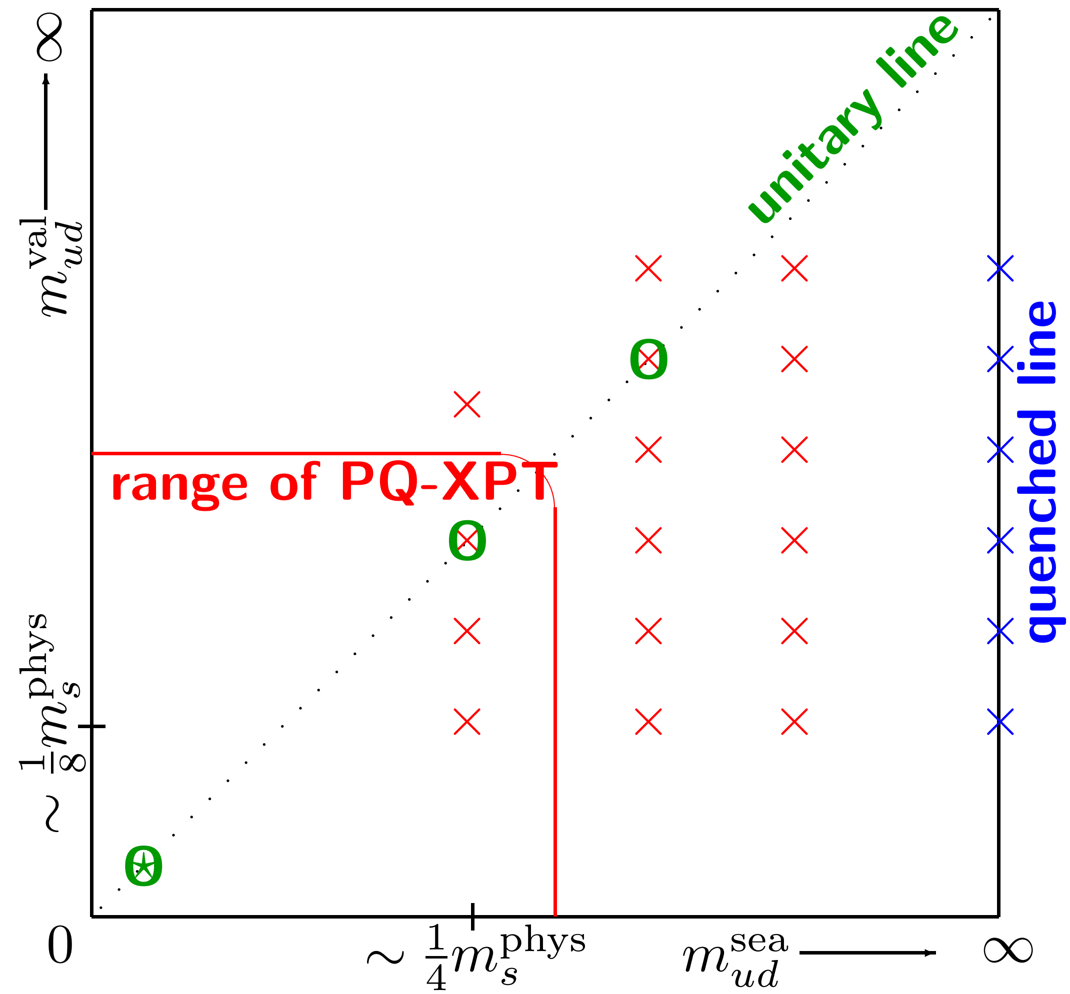

(B) Full QCD

Typically isospin limit $m_u = m_d$ to save CPU time

Allow for $m_{\mathrm{valence}} \neq m_{\mathrm{sea}}$ to fully exploit dynamical ensemble

Strategy of PQ data taking

- PQ-QCD is a useful *extension* of QCD, *same* low-energy constants as (full) QCD; unlike Q-QCD.
- Performing $a \to 0$ first and $m \to 0$ in a second step is safe but requires precise and matched data.
- (crucial) practical issues:
  - **renormalization**
  - **scale setting**
  - **overlap with regime of XPT**



Q/PQ-QCD: Bernard, Goltermann, Sharpe, Colangelo, Pallante, Bijnens, ...

$O(a, a^2)$-effects: Lee, Sharpe, Singleton, Bernard, Aubin, Rupak, Shoresh, Bär, Aoki, ...

Morel representation of Z in Q-QCD and PQ-QCD

Effect of quenching can be described via adding degenerate "bermions", i.e. bosonic/commuting/ghost spin-$\frac{1}{2}$ particles (with otherwise identical quantum numbers)

$$\int d\bar{q}dq\; e^{-\bar{q}(D+m)q} \;=\; \prod_{1}^{N_f} \det(D+m)$$

$$\int d\tilde{q}^\dagger d\tilde{q}\; e^{-\tilde{q}^\dagger(D+m)\tilde{q}} \;=\; \prod_{1}^{N_f} \frac{1}{\det(D+m)}$$

Q-QCD: build $q = (q_1, ..., q_{N_v}|\tilde{q}_1, ..., \tilde{q}_{N_v})^{\mathrm{T}}$ with masses $m_1, ..., m_{N_v}$ in either case.

F-QCD : build $q = (q_1, ..., q_{N_v}, q_1, ..., q_{N_v}|\tilde{q}_1, ..., \tilde{q}_{N_v})^{\mathrm{T}}$ again with same masses.

PQ-QCD: build $q = (q_1, ..., q_{N_v}, q_1, ..., q_{N_s}|\tilde{q}_1, ..., \tilde{q}_{N_v})^{\mathrm{T}}$ where N_s may differ from N_v and pertinent masses are usually different (even if $N_v = N_s$); define $\bar{q} = (\bar{q}_1, ...|\tilde{q}_1^\dagger, ...)$.

Naively, when $M \to 0$, have graded version of QCD chiral symmetry:

$q_{L,R} \to V_{L,R}\, q_{L,R}$ and $\bar{q}_{L,R} \to \bar{q}_{L,R} V_{L,R}^\dagger$ with $V_{L,R} \in U(N_v + N_s|N_v)$

Apparent symmetry is: $SU(N_v + N_s|N_v)_L \times SU(N_v + N_s|N_v)_R \times U(1)_V$

Graded groups

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$ with A, D having commuting ("bosonic") entries; B, C having anti-commuting ("fermionic") entries

Unitary $UU^\dagger = U^\dagger U = 1$; special means $\mathrm{sdet}(U) = \det(A - BD^{-1}C)/\det(D) = 1$; $\mathrm{str}(U) = \mathrm{tr}(A) - \mathrm{tr}(D)$; logarithm through $U = \exp(\mathrm{i}\sqrt{2}\phi/F)$.

Symmetry breaking pattern for PQ-XPT

Conjectured SSB: $SU(N_v + N_s|N_v)_L \times SU(N_v + N_s|N_v)_R \longrightarrow SU(N_v + N_s|N_v)_V$

Highly non-trivial (but testable) hypothesis on the dynamics of the theory !

With $U = \exp(\mathrm{i}\sqrt{2}\phi/F)$ and graded hermitean supertrace-free ϕ the rest is standard [up to a subtlety in the ghost sector and fewer Cayley-Hamilton relations].

Note the enormous cancellation between Goldstone bosons and Goldstone fermions: Goldstone bosons can be $q\bar{q}$ or $\tilde{q}\tilde{q}^\dagger$ states. Goldstone fermions are $\tilde{q}\bar{q}$ or $q\tilde{q}^\dagger$ states.

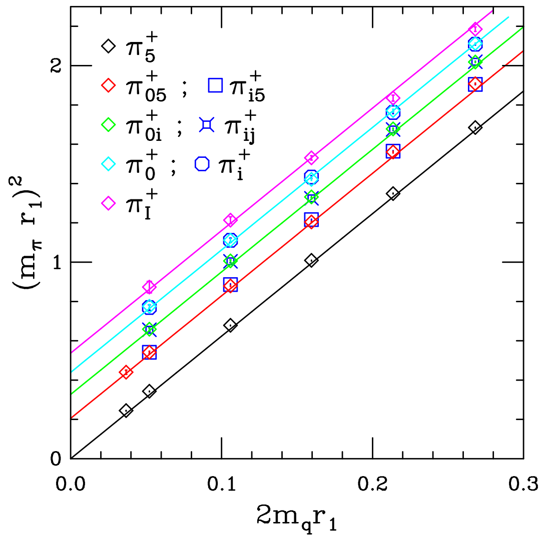
SSB of $SU(N_v + N_s|N_v)_A$ leads to $(N_v + N_s)^2 + N_v^2 - 1$ bosons and $2N_v(N_v + N_s)$ fermions. Net effect equivalent to a theory with $N_s^2 - 1$ bosons — just as desired !

Lattice extension: cut-off effects

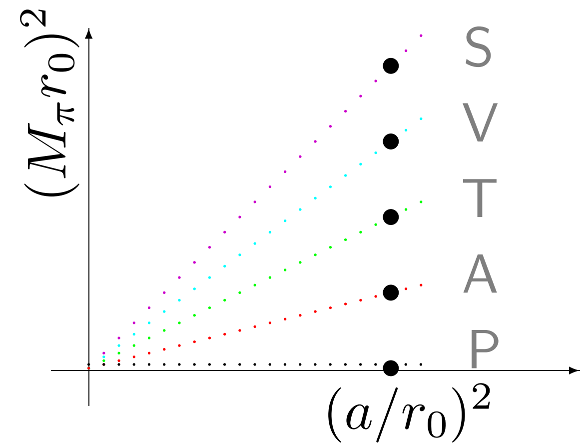
- Wilson fermions: break chiral symmetry, since $\bar{\psi} D_{\mathrm{W}} \psi = \bar{\psi}(\frac{\nabla+\nabla^*}{2} + ... - \frac{a}{2}\nabla_\mu \nabla_\mu^*)\psi$
- clover fermions: ditto, but milder
- near-chiral fermions [fat-link/planar/chirally-improved]: ditto, even milder
- chiral fermions [overlap/domain-wall with $L_5 \to \infty$]: $\gamma_5 D + D\gamma_5 = \frac{a}{\rho} D\gamma_5 D$ [GW]
- perfect fermions: again GW relation, continuum-like dispersion relation
- twisted-mass fermions: mass term is rotated into $V_R M V_L^\dagger$ (modulo renormalization)
- staggered fermions: 4 near-degenerate species per continuum flavor

Extend continuum XPT to account for formulation-specific cut-off effects. Crucial for staggered, desirable for twisted-mass fermions, as they have unphysical isospin-violations. Useful for non-chiral actions to connect cut-off effects in different observables.

Staggered XPT



Taste splitting makes most $\bar{d}(\gamma_5 \otimes T)u$ combinations become non-Goldstone bosons:



Assumption: With N_f flavors of (4-taste) quark fields the pattern of SSB is $SU(4N_f)_L \times SU(4N_f)_R \rightarrow SU(4N_f)_V$ leading to $16N_f^2-1$ pseudo-Goldstone bosons, collected in the 12×12 matrix ($N_f=3$)

$$U = e^{\mathrm{i}\Phi/f} \quad \Phi = \begin{pmatrix} \Phi_u & \pi^+ & K^+ \\ \pi^- & \Phi_d & K^0 \\ K^- & \bar{K}^0 & \Phi_s \end{pmatrix} = \sum_{a,b=1}^{9,16} \Phi^{ab}\frac{\lambda^a}{2}T^b \quad M = \begin{pmatrix} m_u I_4 & 0 & 0 \\ 0 & m_d I_4 & 0 \\ 0 & 0 & m_s I_4 \end{pmatrix}$$

with $U \rightarrow V_R U V_L^\dagger$ under chiral rotations. Counting $p^2 \sim m \sim a^2$ yields

$$\mathcal{L} = \frac{f^2}{8}\mathrm{tr}(\partial_\mu U \partial_\mu U^\dagger) - \frac{\Sigma}{2}\mathrm{tr}(MU+MU^\dagger) + \frac{2m_0^2}{3}(\Phi_{u,\mathrm{TS}}+\Phi_{d,\mathrm{TS}}+\Phi_{s,\mathrm{TS}})^2 + a^2 V_{\mathrm{TB}}$$

Summary

QCD is a fascinating field theory – many detailed predictions from one Lagrangian !

- Conventional PT fails for low-energy observables (coupling is strong).
- Lattice QCD solves the problem by brute force, often interesting regime accessible only via extrapolation ($m \to 0$, $V \to \infty$, $q^2 \to 0$, ...).
- XPT helps, since it knows about collective degrees of freedom ($\pi, K, \eta, ...$) and about their soft interaction at low relative momentum:
  - Goldstone theorem provides understanding of soft-coupling low-energy ds.o.f.
  - symmetries of underlying Lagrangian dictate setup of effective low-energy theory
  - dimensional regularization allows to use conventional weak-coupling techniques
  - wealth of relations among otherwise unrelated low-energy processes (exp & lat)
- Dedicated extensions of XPT for partial quenching and lattice artefacts rest on non-trivial (but testable) assumptions for the SSB pattern, rest is standard.
- Progress likely from combining LQCD & XPT & other fields: plenty of work for you!

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Glossary

Q-QCD:	quenched QCD [determinant excluded]
F-QCD:	full (ordinary) QCD [determinant included]
PQ-QCD:	partially quenched QCD [determinant has “wrong” quark masses]
PQCD:	perturbative QCD
LQCD:	lattice QCD
XPT:	chiral perturbation theory
FGL:	Fritzsch-GellMann-Leutwyler
GOR:	GellMann-Oakes-Renner
GO:	GellMann-Okubo
GL:	Gasser-Leutwyler
GW:	Ginsparg-Wilson
WW:	Wigner-Weyl
NG:	Nambu-Goldstone
LA:	Lie algebra
SSB:	spontaneous symmetry breaking/breakdown
EOM:	equation of motion
RGI:	renormalization group invariant