Finite-volume corrections for masses and decay constants

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based on work in collaboration with

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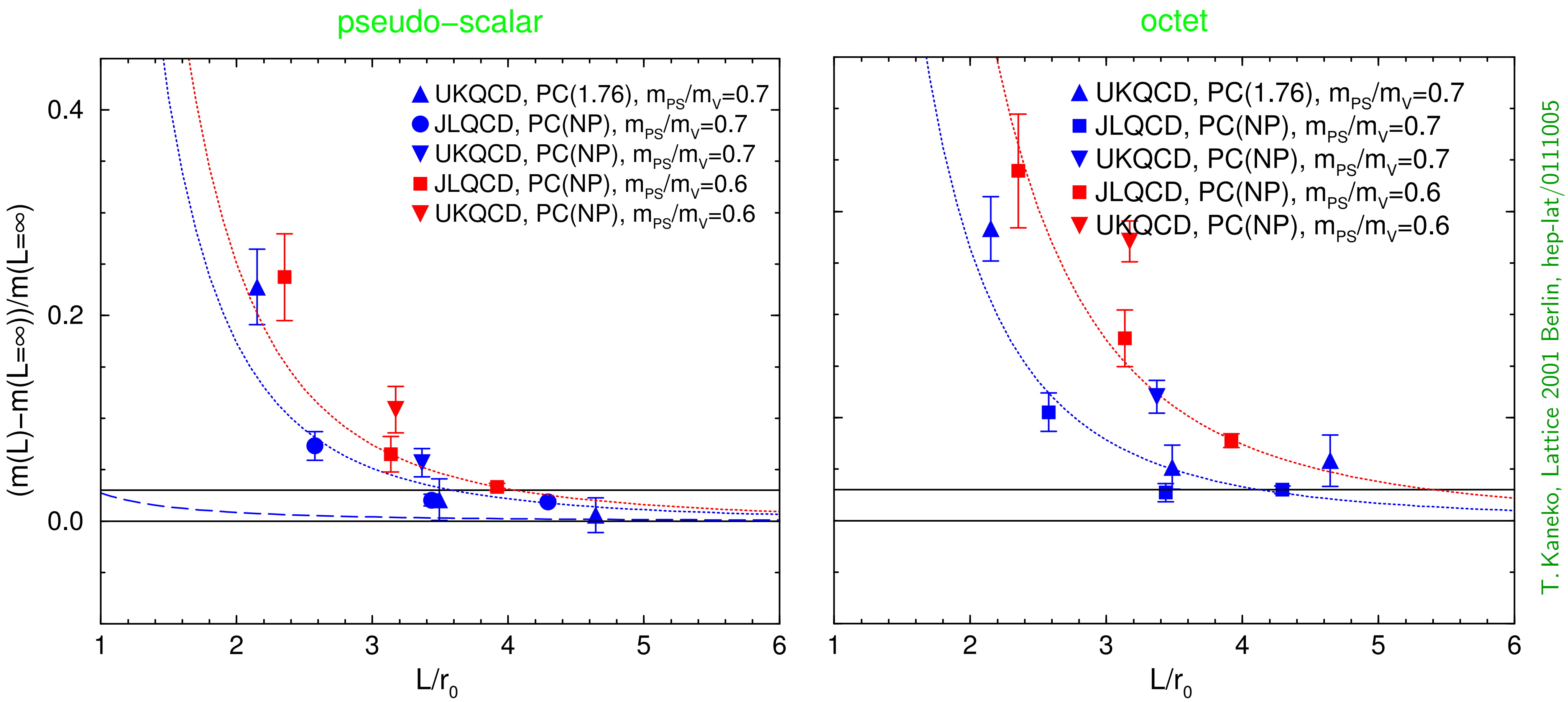
LHP workshop, JLAB Newport News VA, Jul 31 - Aug 3 2006

Finite volume $V = L^3$ affects mass/matrixelements of correlators $C(t)$ with $T \to \infty$:



in this talk:

- exponential correction $M_{\mathrm{had}}(L)/M_{\mathrm{had}}(\infty)$ can be calculated in EFT
- whenever result matters a 1-loop XPT calculation is not sufficient

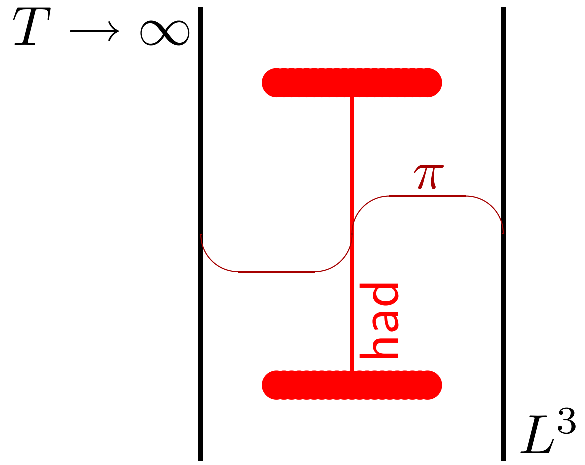
Overview

- EFT calculations of finite-volume correction factors
- Elements of XPT in infinite volume
- Chiral counting: p-regime versus ϵ/δ-regimes
- Straightforward XPT versus Lüscher formula
- Application: $M_\pi(L)/M_\pi$ to (approximate) 3-loop order
- Application: $F_\pi(L)/F_\pi$ to (approximate) 2-loop order
- Comment: $B_K(L)/B_K$ to (approximate and full) 1-loop order
- Comment: $M_p(L)/M_p$ to (approximate and full) 1-loop order

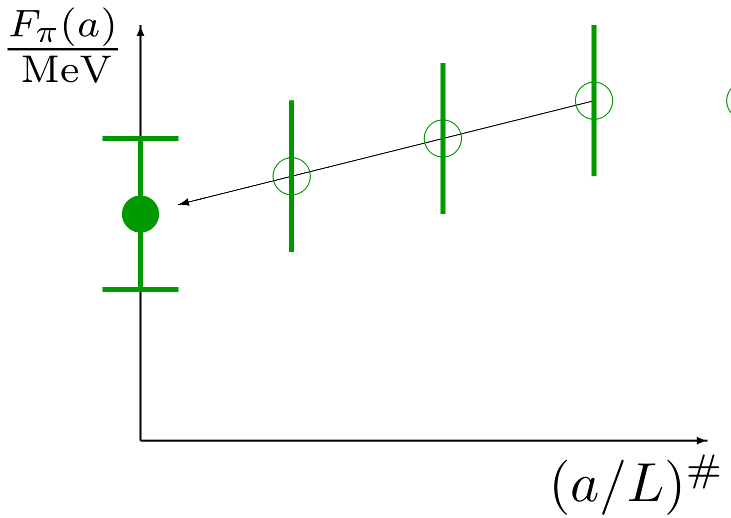
EFT calculations of finite-volume correction factors

idea: $$M_{\rm had}(\infty) = \underbrace{M_{\rm had}(L)}_{\text{lattice}} \cdot \underbrace{\frac{M_{\rm had}(\infty)}{M_{\rm had}(L)}}_{\text{EFT}}$$

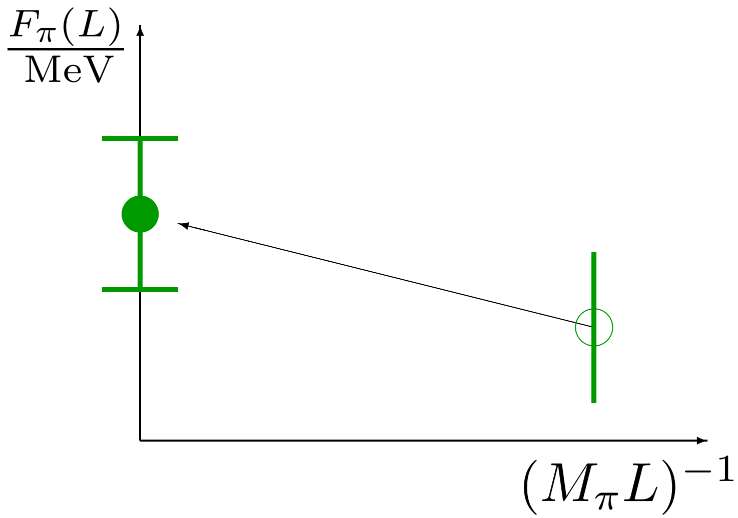
core: $$\frac{M_{\rm had}(\infty)}{M_{\rm had}(L)} = 1 - \mathrm{const}\; e^{-M_\pi L} \quad (\forall\, \mathrm{had})$$



- EFT by Symanzik, ...



- EFT by Gasser, Leutwyler, ...



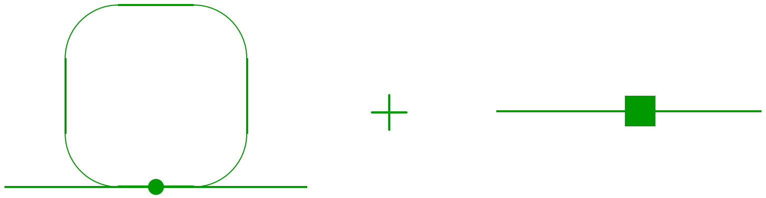
◇ UV-physics (from "cut-off effects")
◇ specific "high-energy constants" (action)
◇ functional form guides extrapolation

◇ IR-physics (from "around the world")
◇ univeral "low-energy constants" (QCD)
◇ one-step correction of single datapoint

Elements of XPT in infinite volume

XPT: $\mathcal{L} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \ldots$ expansion in $p^2 \sim m$

$\Sigma_\pi^{\rm NLO} =$



Contributions to pion self-energy at NLO: 1-loop graph with a vertex from $\mathcal{L}^{(2)}$ [tiny dot] and a counterterm from $\mathcal{L}^{(4)}$ [fat box]. The divergent parts ($\propto \epsilon^{-1}$) compensate each other, and in the finite parts ($\propto \epsilon^0$) the μ-dependence cancels exactly.

$\longrightarrow$ all interactions are parity even and involve (an even number of) derivatives

$\longrightarrow$ theory only order-by-order renormalizable

$\longrightarrow$ results depend on $m, F, B, \Lambda_i, \ldots$ not on μ

LO: $M_\pi^2 = 2mB \equiv M^2$ with $m = \frac{m_u + m_d}{2}$

NLO: $M_\pi^2 = M^2\{1 - \frac{M^2}{32\pi^2F^2}\log(\frac{\Lambda_3^2}{M^2})\}$ with $F = \lim_{m\to 0} F_\pi$

NNLO: $M_\pi^2 = M^2\{1 - \ldots + \frac{M^4}{256\pi^4F^4}[\frac{17}{8}\log^2(\frac{\Lambda_M^2}{M^2}) + k_M]\}$

with $\Lambda_3 = 0.6 \pm^{1.4}_{0.4}\,\mathrm{GeV},\ \Lambda_M = 0.6 \pm 0.03\,\mathrm{GeV},\ k_M = 0 \pm 2$

Attention: use $F_\pi = f_\pi/\sqrt{2} = 130\,\mathrm{MeV}/\sqrt{2} = 92\,\mathrm{MeV}$ at $M_\pi = 140\,\mathrm{MeV}$

Chiral counting: p-regime versus ϵ/δ-regimes

In finite (spatial) volume $V = L^3$ only momenta $\vec{p} = \frac{2\pi}{L}\vec{n},\ \vec{n} \in \mathbf{Z}^3$ possible

Two basic conditions for XPT in finite volume: $(\Lambda_{\rm XPT} \simeq 4\pi F_\pi \simeq 1\,\mathrm{GeV})$

(1) $m \ll \Lambda_{\rm XPT}$ or $M_\pi \ll 4\pi F_\pi$ (2) $\dfrac{2\pi}{L} \ll \Lambda_{\rm XPT}$ or $1 \ll 2F_\pi L$

Once satisfied, still two varieties for pion correlation length:

(3a) $M_\pi L \gg 1$: $M_\pi^2 \sim \frac{1}{L^2} \sim m$ "p-regime" for $T \to \infty$

(3b) $M_\pi L \leq 1$: $M_\pi^2 \sim \frac{1}{L^4} \sim m$ "ϵ-regime" for $T \to \infty$

Physics of p- and ϵ-regimes very much different:

- "p-regime": exponentially small finite-volume corrections
- "ϵ-regime": global pion-field zero-mode needs exact treatment

$$M_\pi(m=0, L \gg \frac{1}{2F}) \sim \frac{{N_f}^2 - 1}{N_f F^2 L^3}$$

Remainder of this talk $\begin{cases} \text{full QCD with } N_f = 2 \text{ or } N_f = 2+1 \\ p\text{-regime with } M_\pi^2 \sim \frac{1}{L^2} \sim m \text{ counting} \end{cases}$

◇ Setup for XPT in finite volume with periodic boundary conditions [Gasser Leutwyler 1987]

Lagrangian: $\mathcal{L}_{\rm eff}(L) = \mathcal{L}_{\rm eff}(\infty)$

Propagator: $G_L(x^0, \vec{x}) = \sum_{\vec{n} \in \mathbf{Z}^3} G_\infty(x^0, \vec{x} + \vec{n}L)$

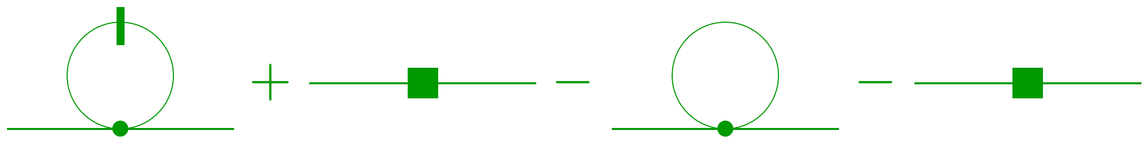
◇ Implication for perturbative calculations (last step: Poisson formula)

$$\int \frac{d^4q}{(2\pi)^4} f(q) \quad \longrightarrow \quad \int \frac{dq^0}{2\pi} \frac{1}{L^3} \sum_{\vec{n} \in \mathbf{Z}^3} f(q^0, \frac{2\pi}{L}\vec{n}) \;=\; \int \frac{d^4q}{(2\pi)^4} f(q) \sum_{\vec{n} \in \mathbf{Z}^3} e^{\mathrm{i}\vec{q}\vec{n}L}$$

Straightforward XPT versus Lüscher formula

- Approach 1: Gasser Leutwyler

$$\Sigma_\pi^{\mathrm{NLO}}(L) - \Sigma_\pi^{\mathrm{NLO}}(\infty) \quad = $$



$\mathcal{L}^{\mathrm{NLO}}$ counterterm drops out and $G_L(x) - G_\infty(x) = \sum_{\vec{n}\in\mathbf{Z}^3\backslash\vec{0}} G_\infty(x^0, \vec{x}+\vec{n}L)$ remains

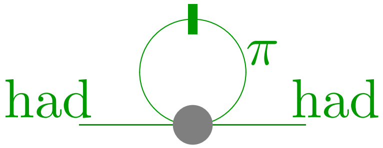
$$\begin{aligned} M_\pi(L) &= M_\pi\left[1 + \frac{1}{2N_f}\xi\tilde{g}_1(\lambda) + O(\xi^2)\right] \\ F_\pi(L) &= F_\pi\left[1 - \frac{2}{N_f}\xi\tilde{g}_1(\lambda) + O(\xi^2)\right] \end{aligned}$$

where $N_f \geq 2$, $M_\pi = M_\pi(\infty)$, $F_\pi = F_\pi(\infty)$, $\xi = \frac{M_\pi^2}{(4\pi F_\pi)^2}$, $\lambda = M_\pi L$ and

$$\tilde{g}_1(\lambda) = \int_0^\infty \sum_{\vec{n}\in\mathbf{Z}^3\backslash\vec{0}} e^{-\frac{1}{\alpha}-\frac{\alpha}{4}\vec{n}^2\lambda^2}\, d\alpha = \sum_{n=1}^{\infty} \frac{4m(n)}{\sqrt{n}\lambda} K_1(\sqrt{n}\lambda)$$

with $m(n)$ the multiplicity of vectors with $|\vec{n}^2| = n$

- Approach 2: Lüscher



$= \int \frac{d^4q}{(2\pi)^4}\,\Gamma(p,q,-q,-p) G_L(q)$ with $\Gamma = \Gamma(\mathrm{had},\pi,\pi,\mathrm{had})$

$$G_\infty(q) \sim \frac{1}{q^2+m^2}\,, \qquad G_L(q) = \sum_{\vec{n}\in\mathbf{Z}^3} G_\infty(q)\, e^{\mathrm{i}\vec{q}\vec{n}L}$$

Asymptotic [large volume, i.e. $\sim e^{-M_\pi L}$] shift comes from one propagator in finite volume

$$\begin{aligned}
M_{\mathrm{had}}(L) - M_{\mathrm{had}} &= \int \frac{d^4q}{(2\pi)^4}\,\Gamma(p,q,-q,-p)\,[G_L(q) - G_\infty(q)] \\
&= \sum_{\vec{n}\in\mathbf{Z}^3\backslash\vec{0}} \int \frac{d^4q}{(2\pi)^4}\,\Gamma(p,q,-q,-p)\,G_\infty(q)\,e^{\mathrm{i}\vec{q}\vec{n}L} \\
&= \ldots \quad \text{[restrict to } \sim e^{-M_\pi L}\text{, perform } \textstyle\int d^3\vec{q}\text{, rename } q^0 = y\text{]} \\
&\simeq \mathrm{const} \int_{-\infty}^{\infty} F(\mathrm{i}y)\, e^{-\sqrt{M_\pi^2+y^2}\,L}\, dy
\end{aligned}$$

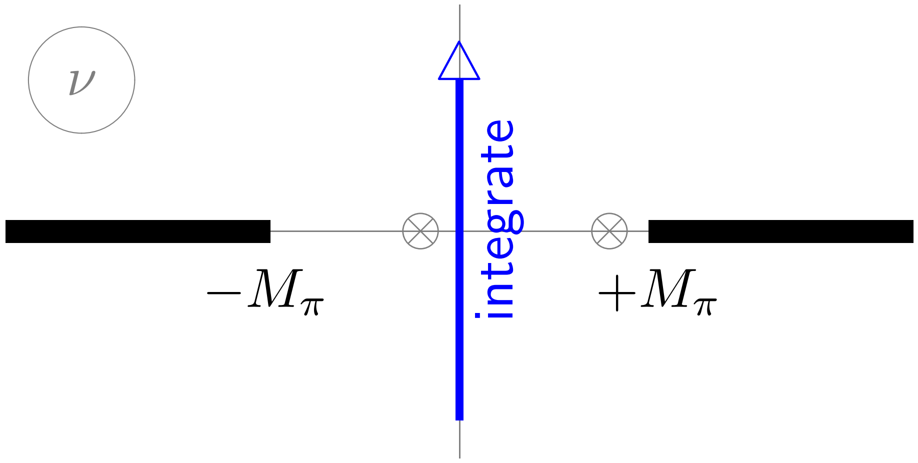
[Do not confuse with 2-particle formula $E^I_{\pi\pi}(L) - 2M_\pi = -\frac{4\pi a_0^I}{M_\pi L^3}(1 + c_1\frac{a_0^I}{L} + c_2(\frac{a_0^I}{L})^2 + \ldots)$]

$$M_{\rm had}(L) - M_{\rm had} = \underbrace{-\frac{6}{32\pi^2 M_{\rm had} L} \int_{-\infty}^{\infty} F(\mathrm{i}y)\, e^{-\sqrt{M_\pi^2+y^2}\,L}\, dy}_{>0} + O(e^{-\sqrt{2}M_\pi L})$$

with forward scattering amplitude

$$F(\mathrm{i}y) = T_{\pi\mathrm{had}}^{I=0}(\nu = \mathrm{i}y, L = \infty)$$

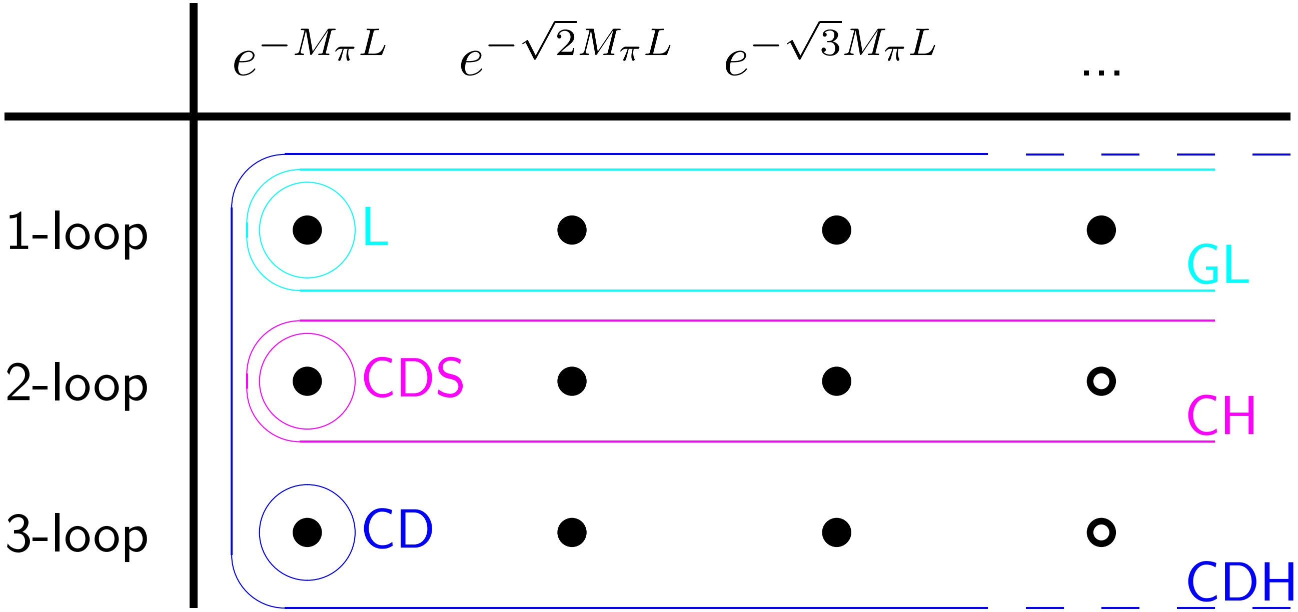
involving $\nu = (s-u)/(4M_{\rm had})$ [Lüscher 1986]



- final result $\sim e^{-M_\pi L}$ for any hadron in (full) QCD
- need $\pi\mathrm{had}$ forward scattering amplitude away from cuts (i.e. in unphysical region)
- in practice invoke XPT in infinite volume to do analytic continuation
- still gain 1 loop order [Cutkosky argument !]
- additional poles on the l.h.s. give extra terms [nucelon], those on the r.h.s. do not

- Twofold expansion

chiral order [1-loop, 2-loop, 3-loop, ...] & large volume [$e^{-M_\pi L}$, $e^{-\sqrt{2}M_\pi L}$, $e^{-\sqrt{3}M_\pi L}$, ...]



Example: $M_\pi(L)/M_\pi$

L: Lüscher 1986
GL: Gasser Leutwyler 1987
CDS: Colangelo Dürr Sommer 2002
CD: Colangelo Dürr 2003
CDH: Colangelo Dürr Haefeli 2005
CH: Colangelo Haefeli 2006

- Resummed Lüscher formula

$$M_{\rm had}(L) - M_{\rm had} = -\frac{1}{32\pi^2 M_{\rm had} L} \sum_{n\geq 1} \frac{m(n)}{\sqrt{n}} \int_{-\infty}^{\infty} F(\mathrm{i}y)\, e^{-\sqrt{(M_\pi^2+y^2)n}\, L}\, dy + O(e^{-\bar{M}L})$$

◇ for the first time used in Colangelo Dürr Haefeli 2005, also $\bar{M} = (\sqrt{3}+1)/\sqrt{2} \cdot M_\pi \simeq 1.93 M_\pi$

◇ estimates higher $e^{-\sqrt{n}M_\pi L}$ contributions, exactly with 0-loop input [reproduces GL 1-loop result], very accurately with 1-loop input Colangelo Haefeli 2006, presumably still so with 2-loop input

Application: $M_\pi(L)/M_\pi$ to (approximate) 3-loop order

Use (orig./res.) Lüscher formula with 0/1/2-loop input to obtain 1/2/3-loop result for

$$R_{M_\pi}(L) \equiv \frac{M_\pi(L) - M_\pi}{M_\pi}$$

- 0-loop input

Invariant amplitude in 2-flavor XPT

$$A(s,t,u)\Big|_{0-\mathrm{loop}} = \frac{s - M_\pi^2}{F_\pi^2} \qquad \longrightarrow \qquad F(\nu)\Big|_{0-\mathrm{loop}} = -\frac{M_\pi^2}{F_\pi^2} \qquad [\nu\text{-independent}]$$

With original formula it follows that [reproduces asymptotic part of 1-loop result by GL]

$$R_{M_\pi} = \frac{6}{16\pi^2 M_\pi L}\frac{M_\pi^2}{F_\pi^2} K_1(M_\pi L) \sim \frac{3}{4(2\pi M_\pi L)^{3/2}}\frac{M_\pi^2}{F_\pi^2} e^{-M_\pi L}$$

With resummed formula it follows that [reproduces complete 1-loop result by GL]

$$R_{M_\pi} = \frac{M_\pi^2}{16\pi^2 F_\pi^2}\sum_{n\geq 1}\frac{m(n)}{\sqrt{n}M_\pi L} K_1(\sqrt{n}M_\pi L)$$

- 1-loop input

◇ Even without resummation the deviation [in parameter regions where it matters] from the 1-loop result is sizable [Colangelo Dürr Sommer 2002]

◇ The resummed Lüscher formula with 1-loop input has been compared to the exact 2-loop result and has been found to be extremely accurate [Colangelo Haefeli 2006]

- 2-loop input

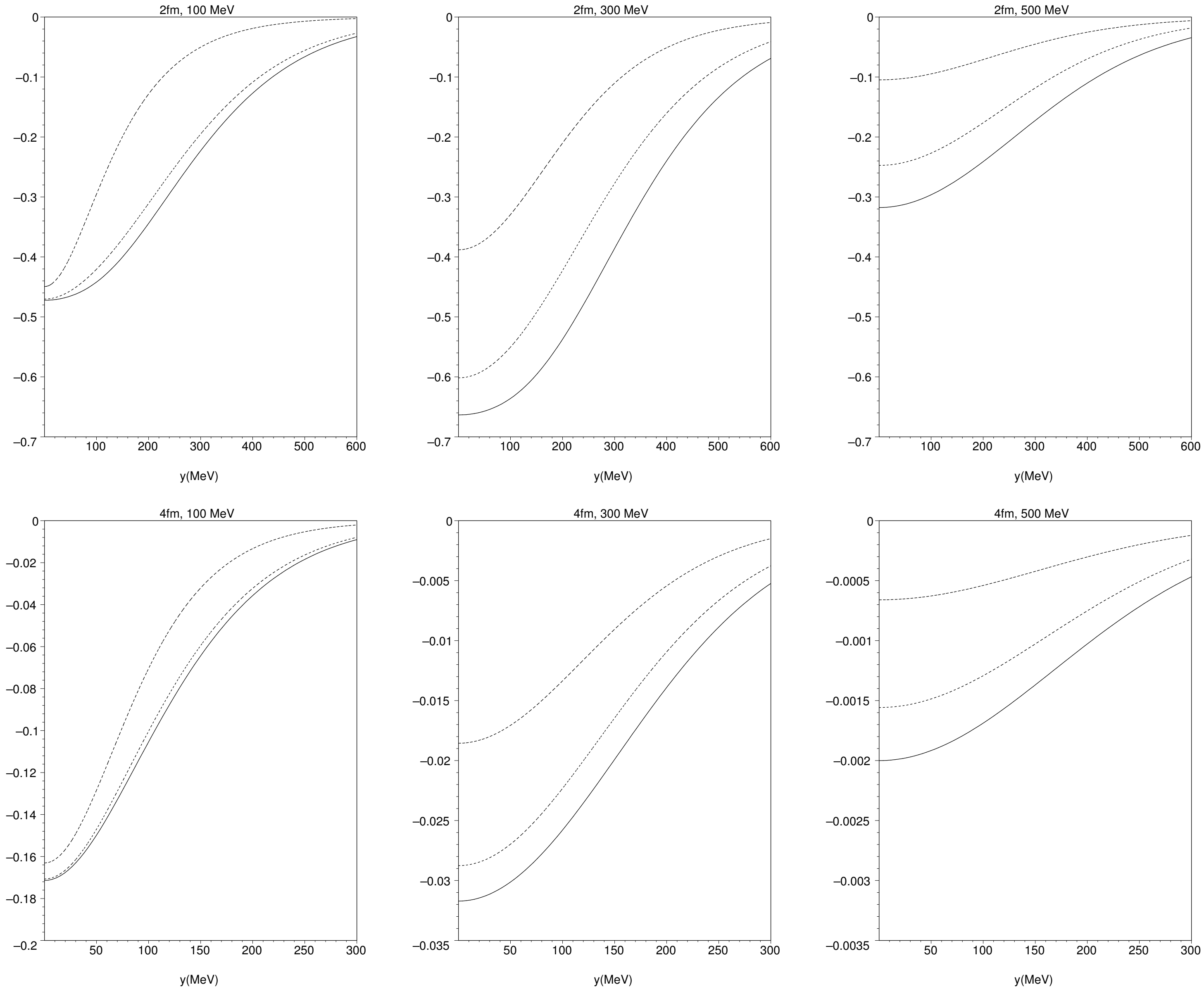
◇ Without resummation one finds good convergence in the chiral/loop order [Colangelo Dürr 2003]

◇ With resummation one obtains the best XPT answer for $M_\pi(L) - M_\pi$ [Colangelo Dürr Haefeli 2005]

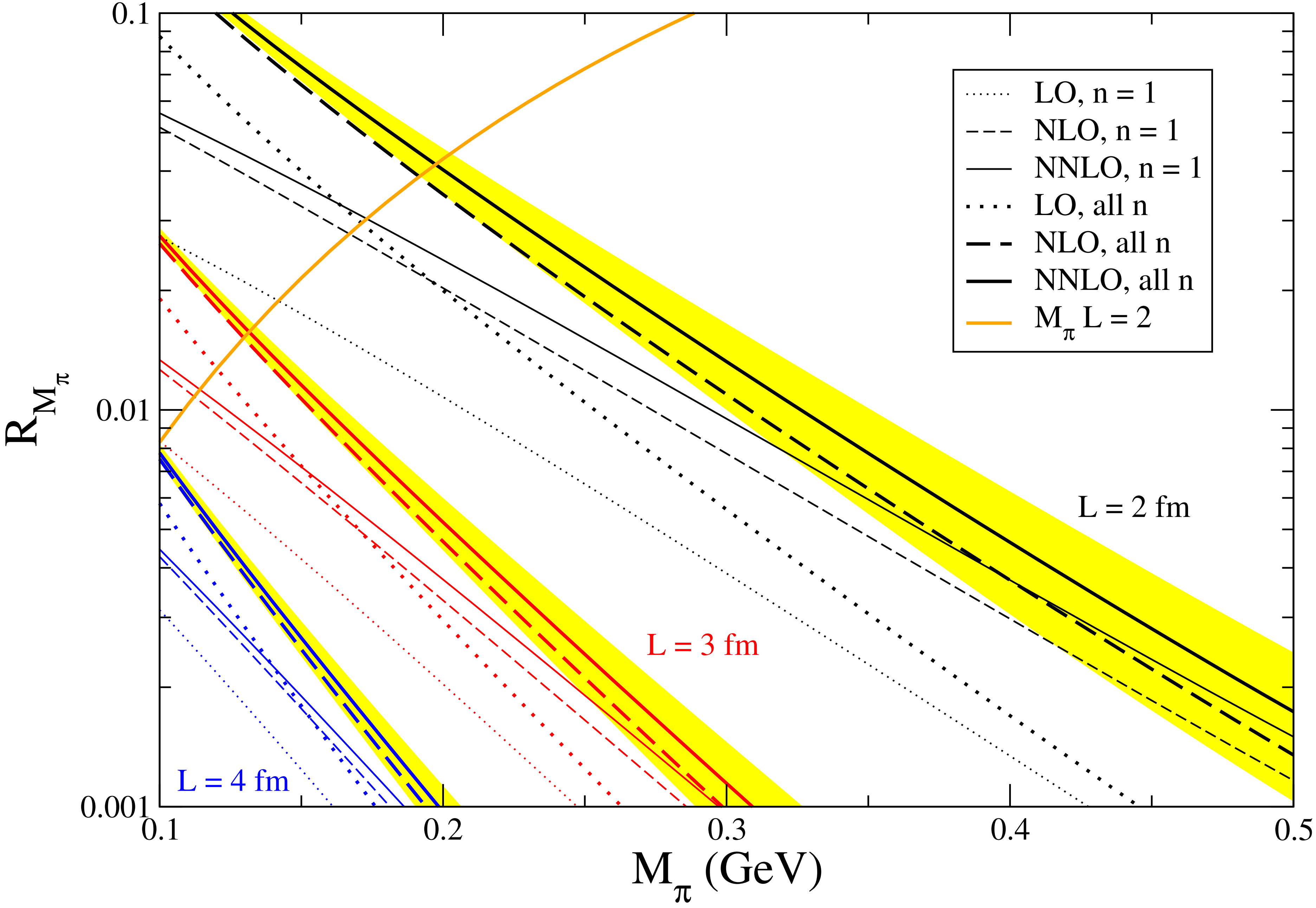
◇ Pertinent NNLO low-energy constants limit precision, precise values for $M_\pi(L) - M_\pi$ would determine new linear combinations [Colangelo Dürr Haefeli 2005]

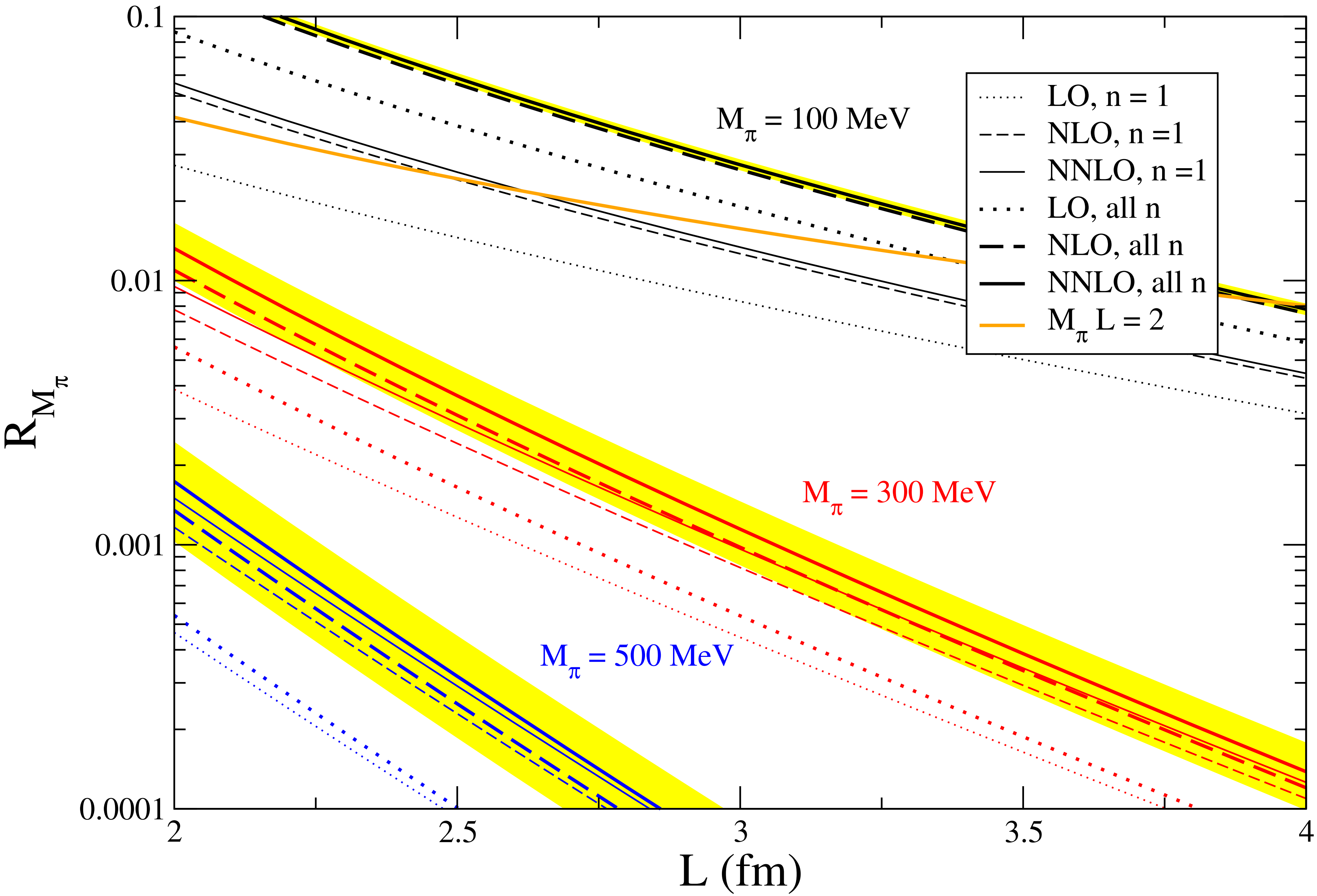
(orig./res.) formula with n-loop input yields (truncated/approximate) $n+1$-loop result

- Assessment of integrand $I(y) = F(\mathrm{i}y)\, e^{-\sqrt{M_\pi^2+y^2}\,L}$ with 0/1/2-loop input

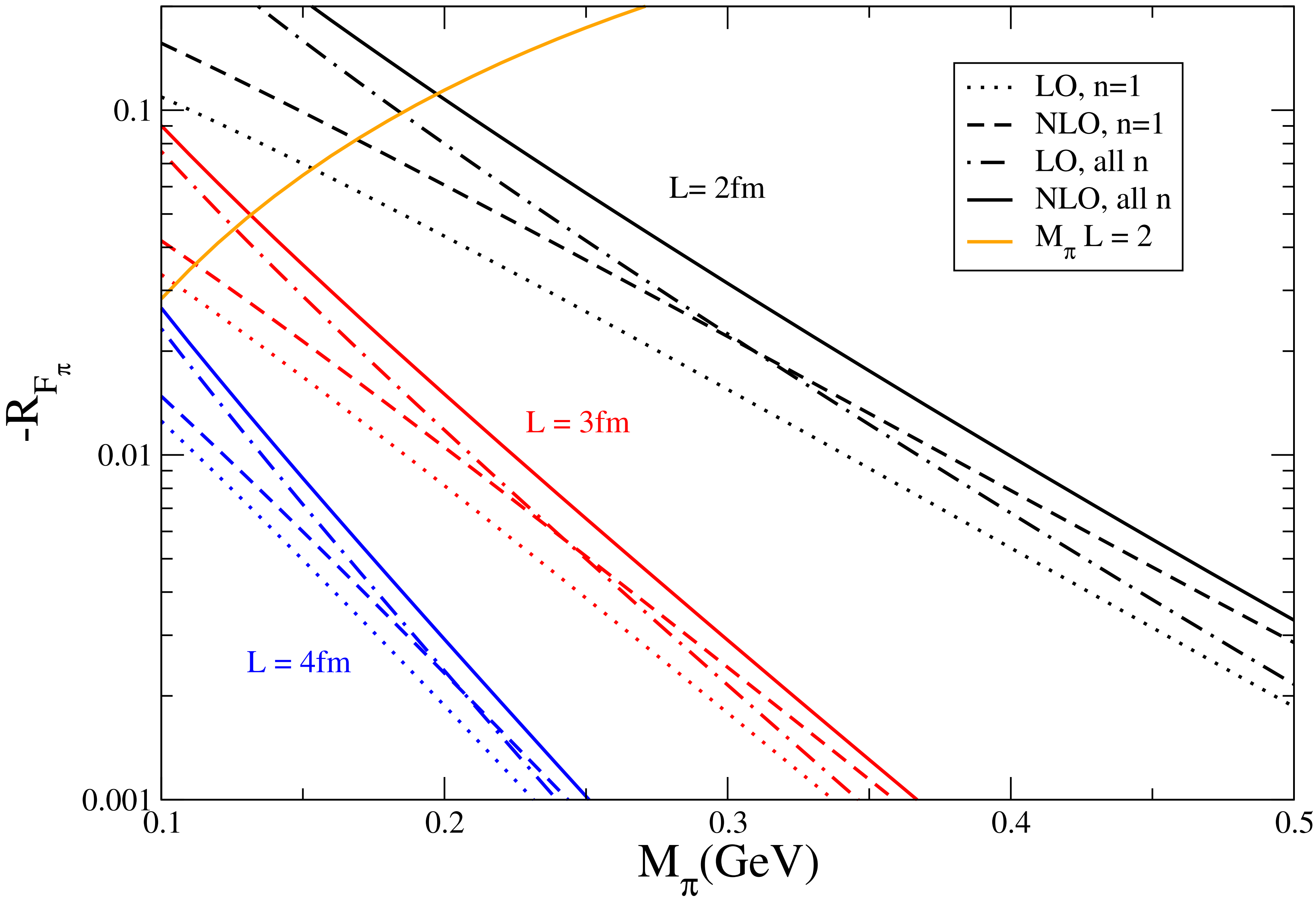


[$M_\pi(L) - M_\pi$ via Lüscher formula – continued]

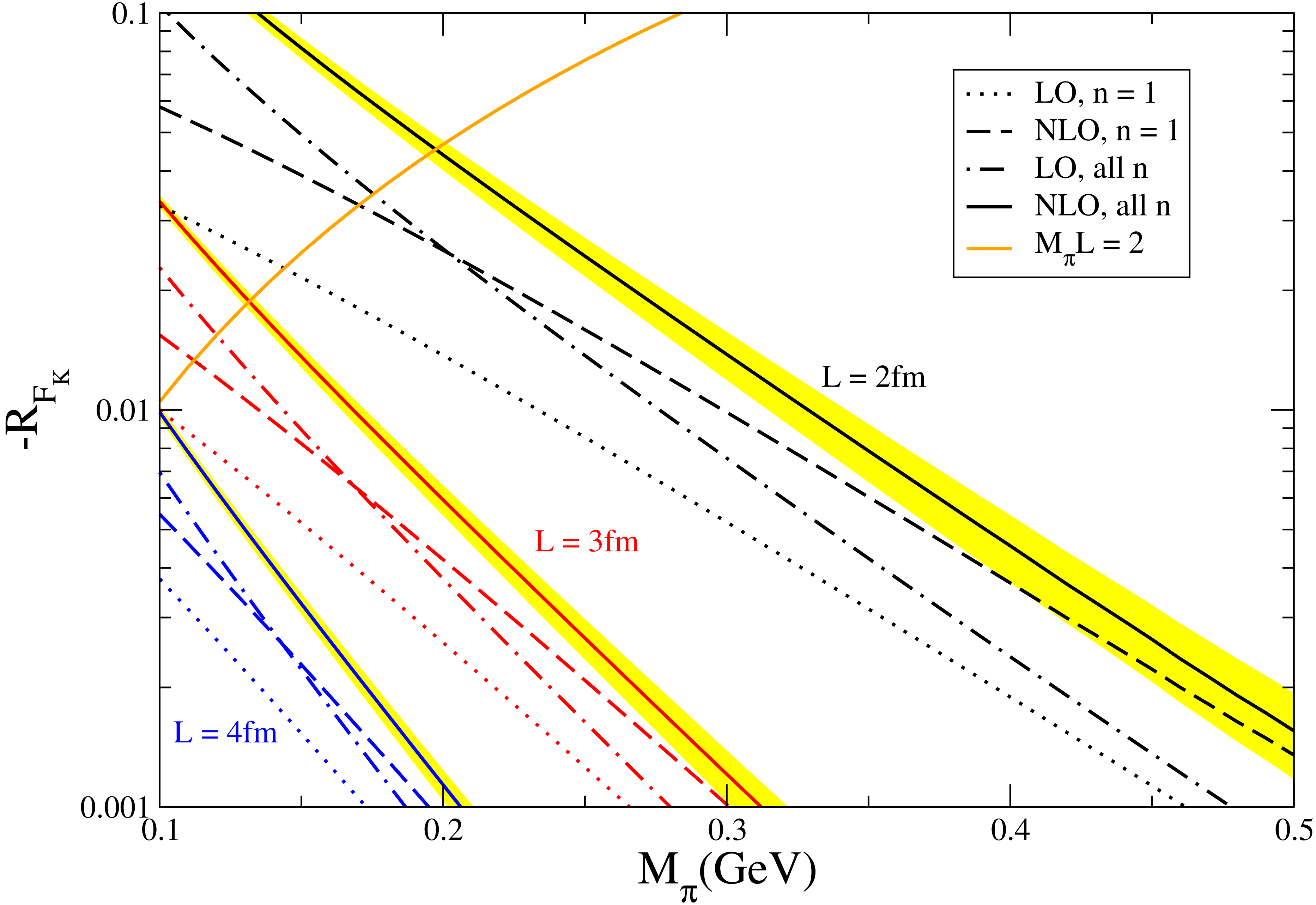



0.1
0.01
0.001
0.0001
2
2.5
3
3.5
4
L (fm)
R_{M_π}
M_π = 100 MeV
M_π = 300 MeV
M_π = 500 MeV
LO, n = 1
NLO, n =1
NNLO, n =1
LO, all n
NLO, all n
NNLO, all n
M_π L = 2

Application: $F_\pi(L)/F_\pi$ to (approximate) 2-loop order



Application: $F_K(L)/F_K$ to (approximate) 2-loop order



Comment: $B_K(L)/B_K$ to (approximate and full) 1-loop order

$$B_K = \frac{\langle \bar{K}^0|\bar{s}\gamma_\mu(1-\gamma_5)d\;\bar{s}\gamma_\mu(1-\gamma_5)d|K^0\rangle}{\frac{8}{3}\langle \bar{K}^0|\bar{s}\gamma_\mu(1-\gamma_5)d|0\rangle\langle 0|\bar{s}\gamma_\mu(1-\gamma_5)d|K^0\rangle}$$

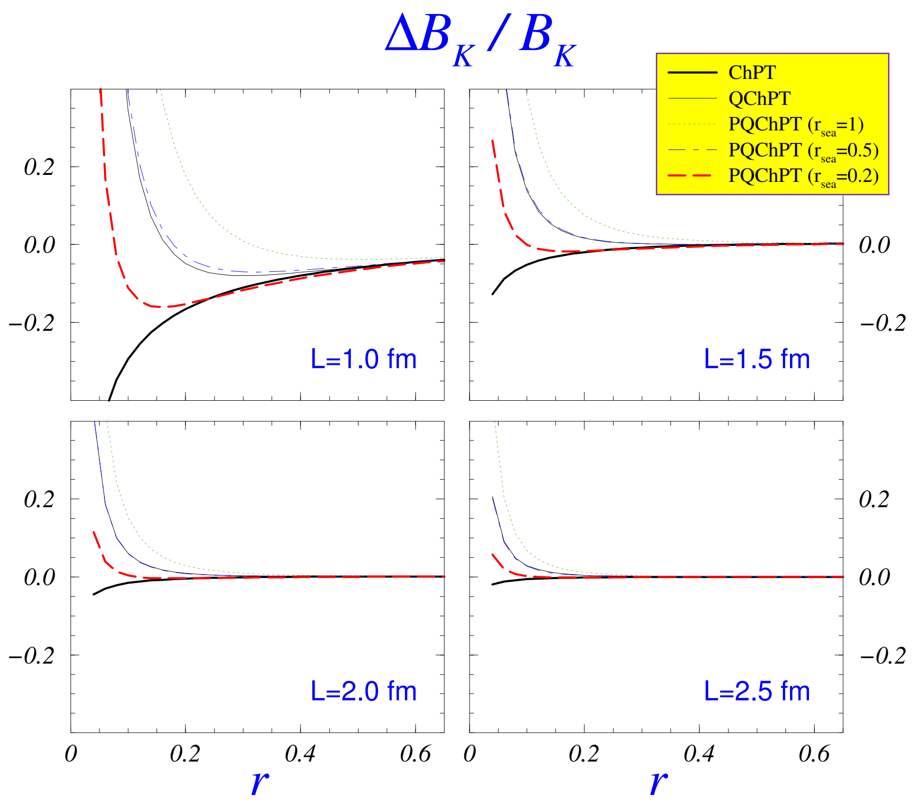
Bećirević and Villadoro give in hep-lat/0311028 1-loop expression for $F_K(L)-F_K$ and $B_K(L)-B_K$ in 3-flavor XPT with/without (partial) quenching.

$$R_{B_K}^{\mathrm{F-QCD}} \simeq -\frac{3}{4}\frac{M_K^2+M_\pi^2}{M_K^2}\Big(\frac{M_\pi}{F_\pi}\Big)^2\frac{e^{-M_\pi L}}{(2\pi M_\pi L)^{3/2}}$$

Plots as a function of $r = m_{ud}/m_s$ indicate

◇ in regime where XPT applicable ($L > 1.5\,\mathrm{fm}$) 1-loop shift in F-QCD small unless $r < 0.15$

◇ sign of R_{B_K} in F-QCD may be different from sign in (P)Q-QCD

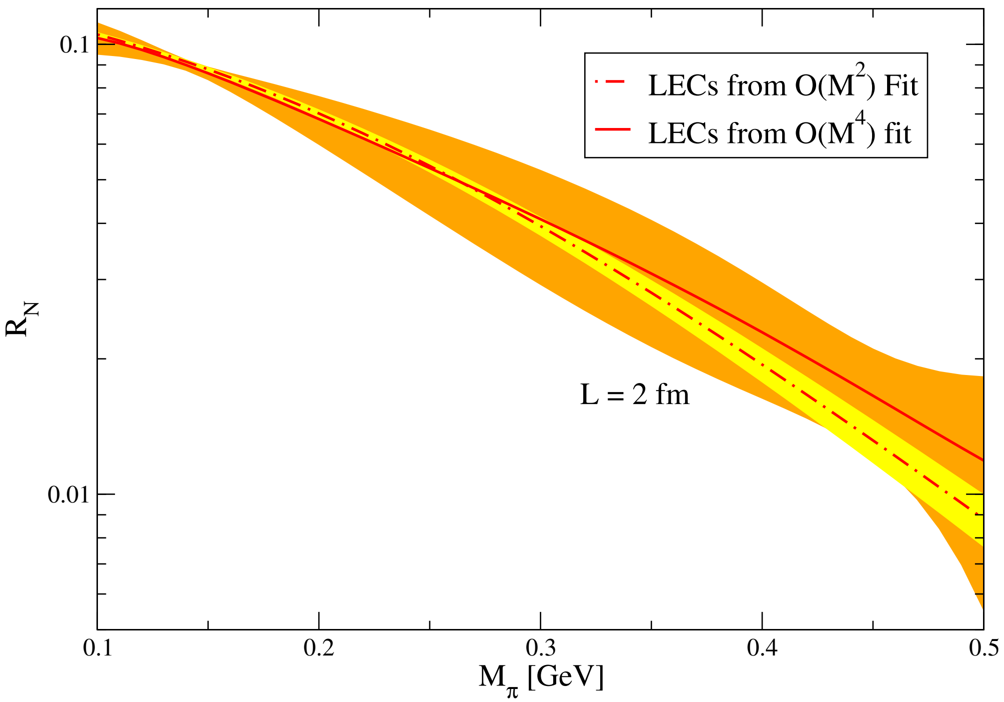


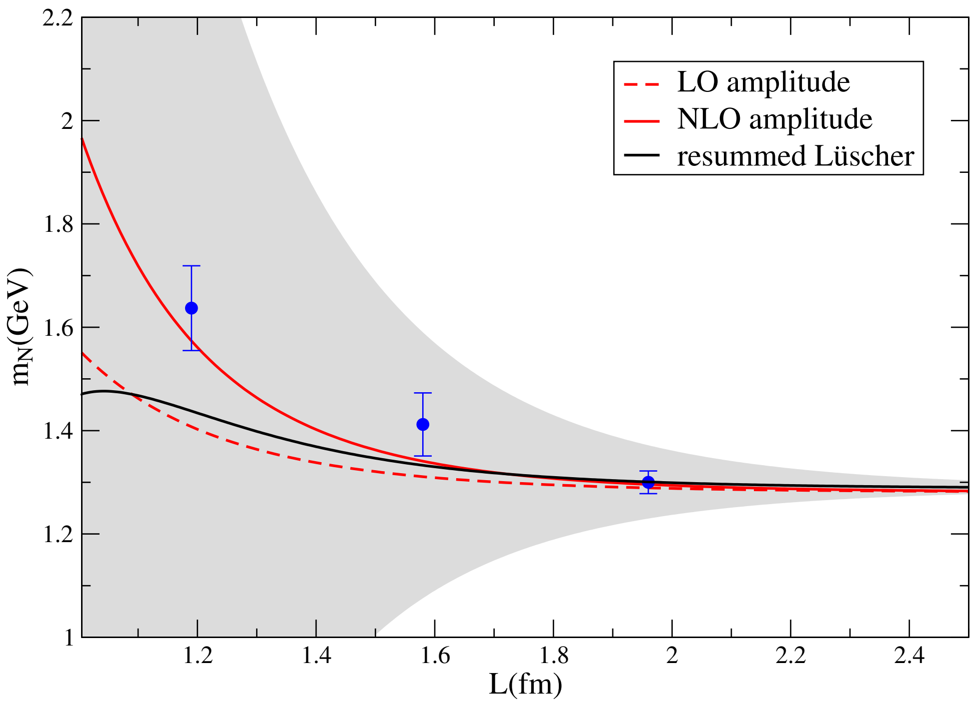
Comment: $M_N(L)/M_N$ to (approximate and full) 1-loop order

$M_N(L)/M_N$ is among the earliest applications of the Lüscher formula.
Careful treatment in AliKhan *et al.* [hep-lat/0312030], Beane [hep-lat/0403015] and Koma^2 [hep-lat/0504009].
Unfortunately, chiral symmetry restricts πN interactions less severely than $\pi\pi$.

$$R_N = \frac{3\epsilon_\pi^2}{4\pi^2} \sum_{n\geq 1} \frac{m(n)}{\sqrt{n}\lambda_\pi} \Big[2\pi\epsilon_\pi g_{\pi N}^2 e^{-\sqrt{n(1-\epsilon_\pi^2)}\,\lambda_\pi} - \int_{-\infty}^{\infty} e^{-\sqrt{n(1+\tilde{y}^2)}\,\lambda_\pi}\, \tilde{D}^+(\tilde{y})\, d\tilde{y} \Big]$$

with $\lambda_\pi = M_\pi L,\ \epsilon_\pi = \frac{M_\pi}{2M_N}$ and $\tilde{D}^+(y) = M_N\, D^+(\mathrm{i}M_\pi y, 0)$.





right: $M_\pi = 545$ MeV

hep-lat/0512002

Summary

- XPT is the proper framework to calculate finite-volume corrections in (full) QCD
- Need $M_\pi \ll 4\pi F_\pi$ and $L \gg (2F_\pi)^{-1} = 1\,\mathrm{fm}$ to apply, formulas for p-regime $[M_\pi L \gg 1]$
- Whenever results matters [i.e. $R > 3\%$] a 1-loop calculation seems insufficient
- Lüscher formula highly economic [input: XPT in $V = \infty$, output: $+1$ loop]
- Test at 2-loop level indicates that resummed version extremely accurate
- Use tables in CDH=hep-lat/0503014 to correct $M_\pi(L) \to M_\pi$ and $F_\pi(L) \to F_\pi$