Theoretical issues with staggered fermion simulations

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based on work together with

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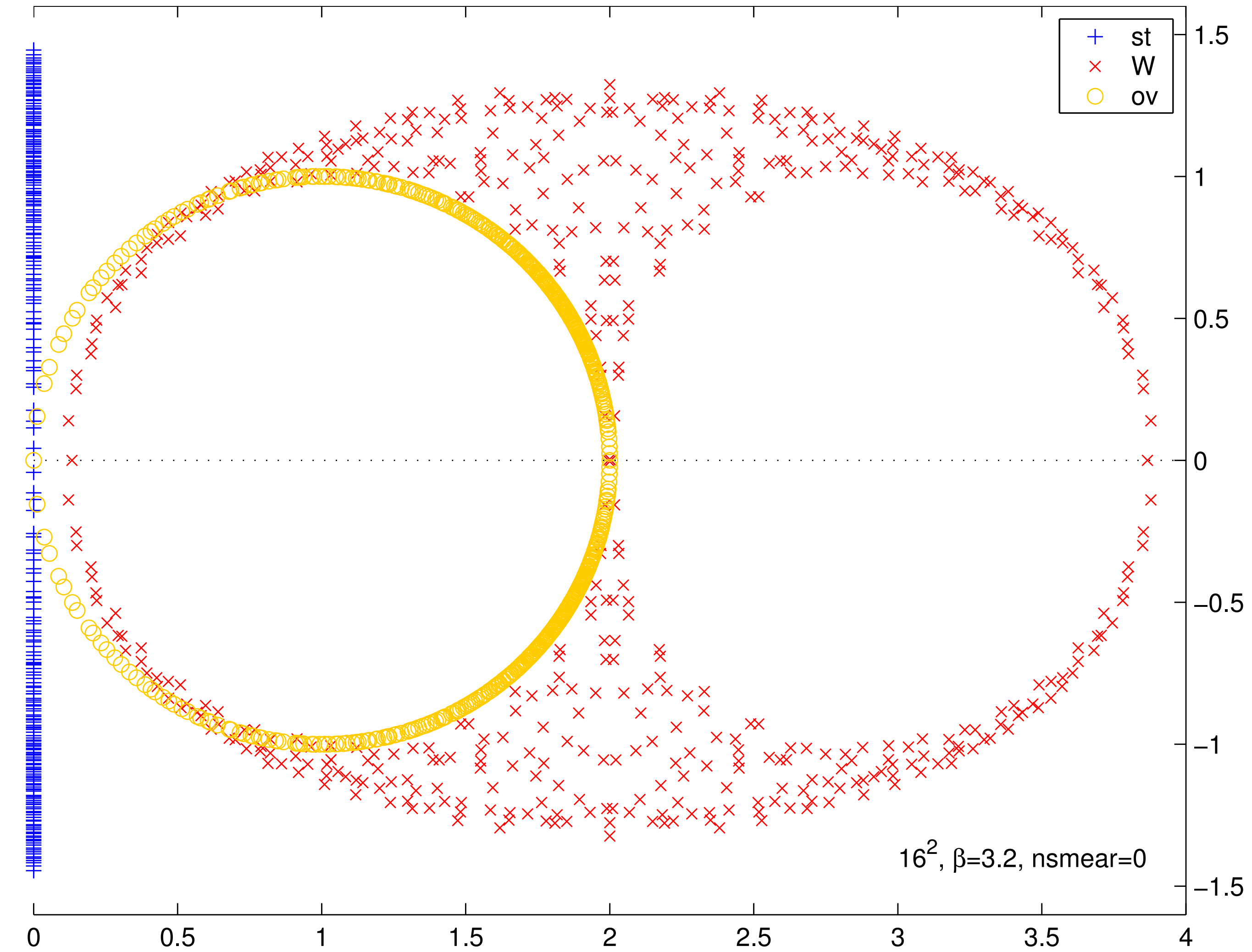
Lattice 05, Dublin, July 25-30, 2005



- $\hat{D} = \mathrm{i}\gamma_\mu p_\mu + O(p^2)$
- $D_{\mathrm{st}}, D_{\mathrm{ov}}$ have no additive mass renormalization, but D_{W} has
- $\gamma_5 D_{\mathrm{W,ov}} \gamma_5 = D^\dagger_{\mathrm{W,ov}}$ and $\eta_5 D_{\mathrm{st}} \eta_5 = D^\dagger_{\mathrm{st}} \Longrightarrow \det(D) \geq 0$
- $\mathrm{spec}(D_{\mathrm{na}}) = \mathrm{spec}(D_{\mathrm{st}}) \otimes I_2$ (would be I_4 in 4D)

Controversy in a nutshell

- In 4D the staggered action yields 4 "tastes" in the continuum, and 3 of these must be excised from ones physical predictions.
- The way how this is done is (in general) different for valence and sea quarks.
- Naively, that difference should leave no trace in the continuum limit.

Question: Is QCD with $N_f = 2+1$ staggered quarks fundamentally correct ?
[Is it physics from first principles or just a (phenomenologically successful) model of QCD ?]

- In the fundamental theory taste splitting may be suppressed through various tricks (improved glue, filtering, RG blocking, ...).
- In the effective theory taste splitting effects may be parametrized and thus "taken away" (approximately) for a variety of observables.

From a conceptual viewpoint either of these improvements is immaterial; if the staggered approach is correct, it yields the right continuum limit for arbitrary observables without any of these.

$\oplus$: Prove that QCD with $N_f = 2+1$ staggered quarks is in the right universality class.
$\ominus$: Find a single observable where the staggered answer, after continuum extrapolation, is wrong.

- **Review: staggered action and taste representation**
- **Problem: rooting versus locality**
- **Free case: four constructions**
- **Interacting theory:** $\mathrm{spec}(D_{\mathrm{st}})$ **in 4D**
- **Interacting theory:** $\chi_{\mathrm{sca}}\,,\chi_{\mathrm{top}}$ **in 2D**
- **Correlation of** $\det^{1/N_t}(D_{\mathrm{st},m})$ **and** $\det(D_{\mathrm{ov},m})$
- **Low-energy unitarity and SXPT**
- **Summary**

Review: staggered action

$$S_{\rm na} = \frac{a^3}{2} \sum_{x,\mu} \bar\psi(x)\,\gamma_\mu\,[U_\mu(x)\psi(x+\hat\mu) - U_\mu^\dagger(x-\hat\mu)\psi(x-\hat\mu)]$$

$\longrightarrow$ Naive action describes 4 (2D) or 16 (4D) fermions (in general: 2^d) in the continuum.

$$\psi(x) = \gamma(x)\chi(x) \qquad \bar\psi(x) = \bar\chi(x)\gamma(x)^\dagger \qquad \gamma(x) = \gamma_1^{x_1}\gamma_2^{x_2}\gamma_3^{x_3}\gamma_4^{x_4} \qquad \eta_\mu(x) = (-1)^{\sum_{\mu<\nu} x_\nu}$$

$$S_{\rm na/st} = \frac{a^3}{2} \sum_{x,\mu} \bar\chi(x)\,\eta_\mu(x)\,[U_\mu(x)\chi(x+\hat\mu) - U_\mu^\dagger(x-\hat\mu)\chi(x-\hat\mu)]$$

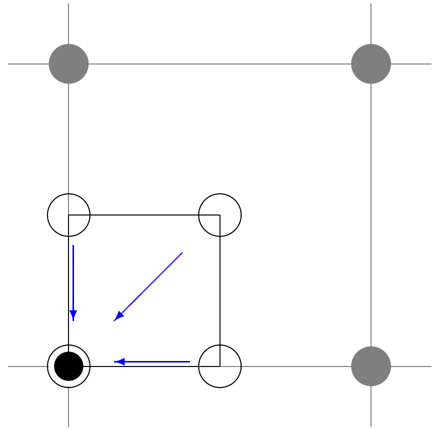
$\longrightarrow$ The 2 (2D) or 4 (4D) components (in general: $2^{d/2}$) decouple.
$\longrightarrow$ Downgrade to one component, i.e. to 2 (2D) or 4 (4D) "tastes" (in general: $N_t = 2^{d/2}$).
$\Longrightarrow$ KS procedure "thins out d.o.f.", but distributes/intertwines spinor and taste.

$$\chi(x) \to e^{i\theta_A\,\eta_5(x)}\chi(x) \qquad \bar\chi(x) \to e^{-i\theta_A\,\eta_5(x)}\bar\chi(x) \qquad (m=0)$$

$\longrightarrow$ Remainder of $SU(2^{d/2})_A$ in taste space is sufficient to forbid additive mass renormalization.
$\Longrightarrow$ Further (exact) "thinning" impossible, since resulting spectrum is non-degenerate.

Review: taste representation

N_f staggered fields $\chi = u, d, s, \ldots$ with 4 tastes each; hypercubic decomposition $\chi(x, x+a\hat{1}, \ldots, x+a\hat{1}+a\hat{2}+a\hat{3}+a\hat{4}) \rightarrow q(X)$ collects $2^{d/2}$ tastes with $2^{d/2}$ components each in one "blocked node".



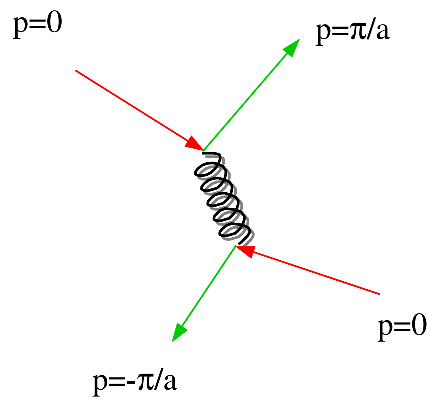
With $\{X\} = \{N\}b$ and $b = 2a$ the free action takes the form

$$S_{\rm st} = b^4 \sum_{X,\mu} \bar{q}(X) \left[\nabla_\mu(\gamma_\mu \otimes I) - \frac{b}{2}\triangle_\mu(\gamma_5 \otimes \tau_\mu \tau_5)\right] q(X)$$

with (spinor$\otimes$taste), $\tau_\mu = \gamma_\mu^*, \tau_5 = \gamma_5$ and the blocked first/second derivative

$$(\nabla_\mu q)(X) = \frac{q(X+b\hat{\mu}) - q(X-b\hat{\mu})}{2b}$$

$$(\triangle_\mu q)(X) = \frac{q(X+b\hat{\mu}) - 2q(X) + q(X-b\hat{\mu})}{b^2} .$$



$\Longrightarrow$ In taste basis the taste interactions stem from a dim=5 Wilson-like term.

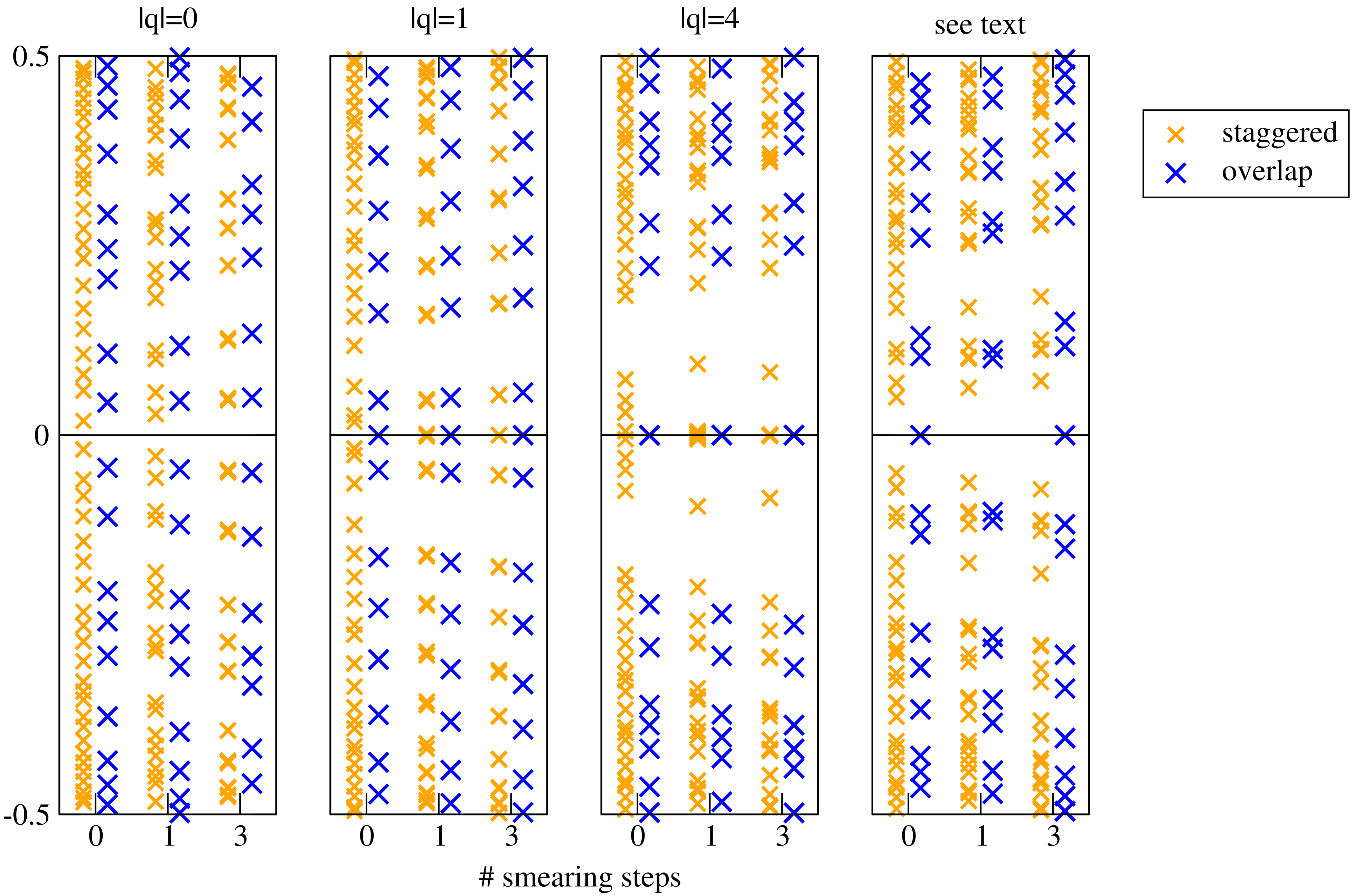
$\longrightarrow$ Do they go away with $a \to 0$ without any trace ? (order of limits?)

Interacting theory:

x-space: different tastes (and individual components of each) see slightly different local gauge field.

p-space: gluons with $p \sim \pi/a$ may kick field from one taste to another (flavor exact with N_f fields).

$\longrightarrow$ Identification staggered=physical flavor may work only, if taste interactions minimized/eliminated.



SD, C.Hoelbling, PRD 69, 034503 (2004) [hep-lat/0311002]

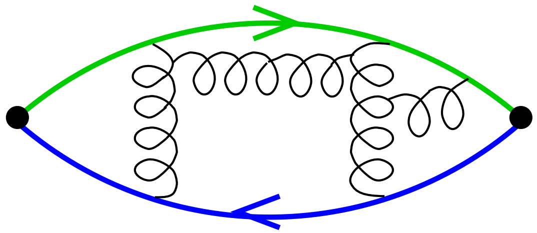
(Schwinger model, $\beta = 4.0, 20^2$)

Problem: rooting versus locality

Marinari,Parisi,Rebbi (1981):
"On the lattice the action $S_G - \frac{1}{4}\mathrm{tr}\,\log(D_{\mathrm{st}})$ will produce a violation of fundamental axioms, but we expect the violation to disappear in the continuum limit and then recover the theory with a single fermion."

- With any undoubled Dirac operator, e.g. $D = D_{\mathrm{W}}$, one has

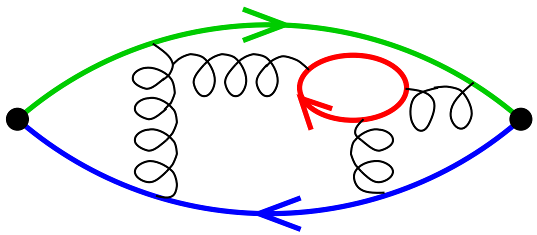
$$Z|_{N_f} = \int D[U,\bar{\psi},\psi]\; e^{-S_G[U]-\sum\int\bar{\psi}D\psi} = \int D[U]\;\det{}^{N_f}(D)\; e^{-S_G[U]}\,.$$



(A) Quenched QCD: quark loops neglected

- With a 4-flavor Dirac operator like D_{st}, one may formally define

$$Z|_{N_f} = \int D[U]\;\det{}^{N_f/4}(D_{\mathrm{st}})\; e^{-S_G[U]}$$



(B) Full QCD

but it is not clear whether there is a 1-taste operator D_{ca} such that

$$\det{}^{1/4}(D_{\mathrm{st}}) = \int D[\bar{\psi},\psi]\; e^{-\int\bar{\psi}D_{\mathrm{ca}}\psi}\,.$$

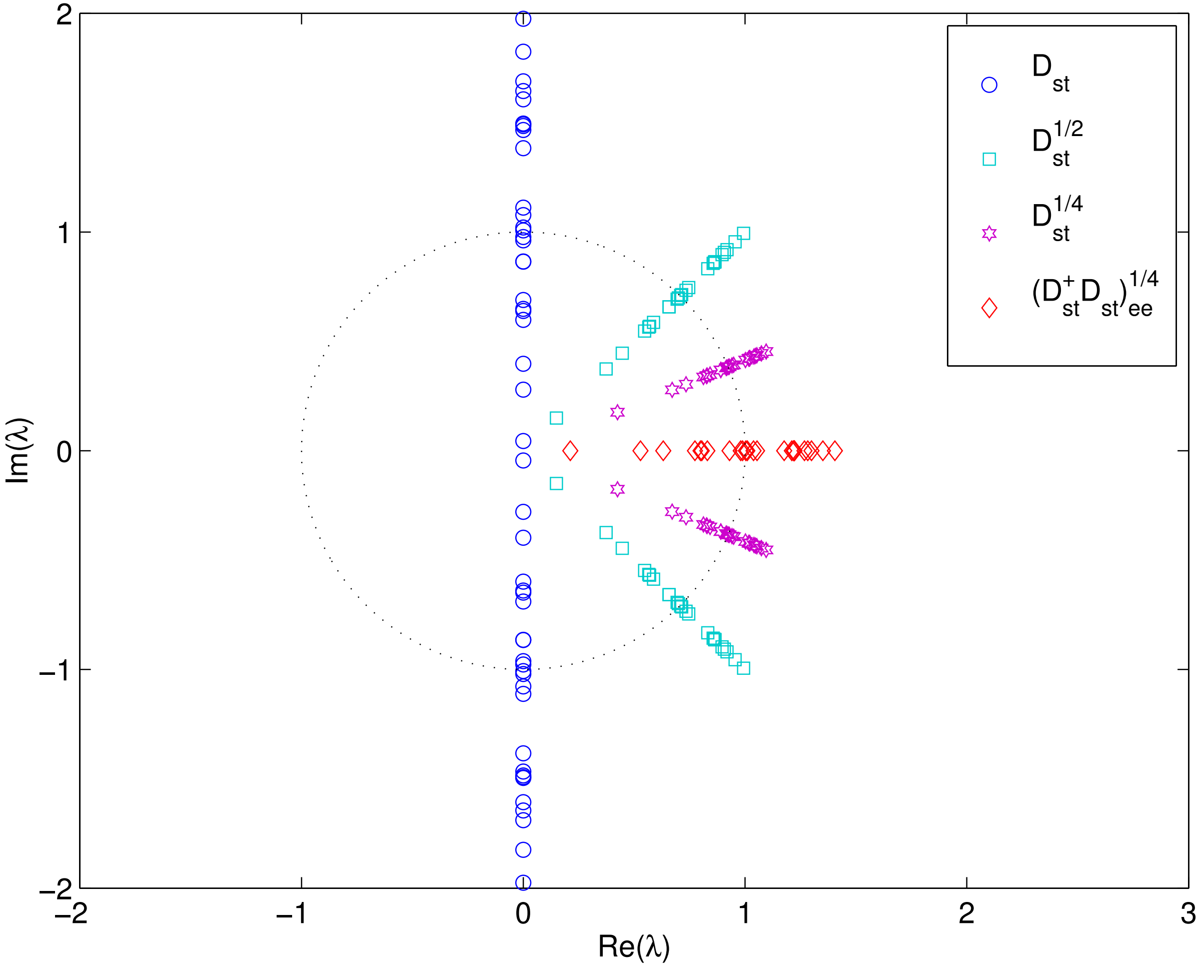
To guarantee locality/causality of arbitrary Green's functions (thus to discuss renormalizability and universality) a "candidate" operator D_{ca} should exist with K.Jansen, NPPS 129, 3 (2004) [hep-lat/0311039]

(1) $\det(D_{\mathrm{ca}}) \xrightarrow{a\downarrow 0} \mathrm{const}\cdot\det{}^{1/4}(D_{\mathrm{st}})$

(2) $||D_{\mathrm{ca}}(x,y)|| < C\, e^{-\nu|x-y|/a}$ with C,ν independent of U.

- **naive rooting of $D_{\rm st}$**

$D_{\rm st}$ is a normal operator ($[D_{\rm st}, D_{\rm st}^\dagger] = 0$), hence $D_{\rm st} = \sum_\lambda \lambda\, \psi(x)\psi^\dagger(y) = U \Lambda U^\dagger$ with U unitary (R/L-eigenvectors in columns) and Λ diagonal, thus $f(D_{\rm st}) = U f(\Lambda) U^\dagger$.



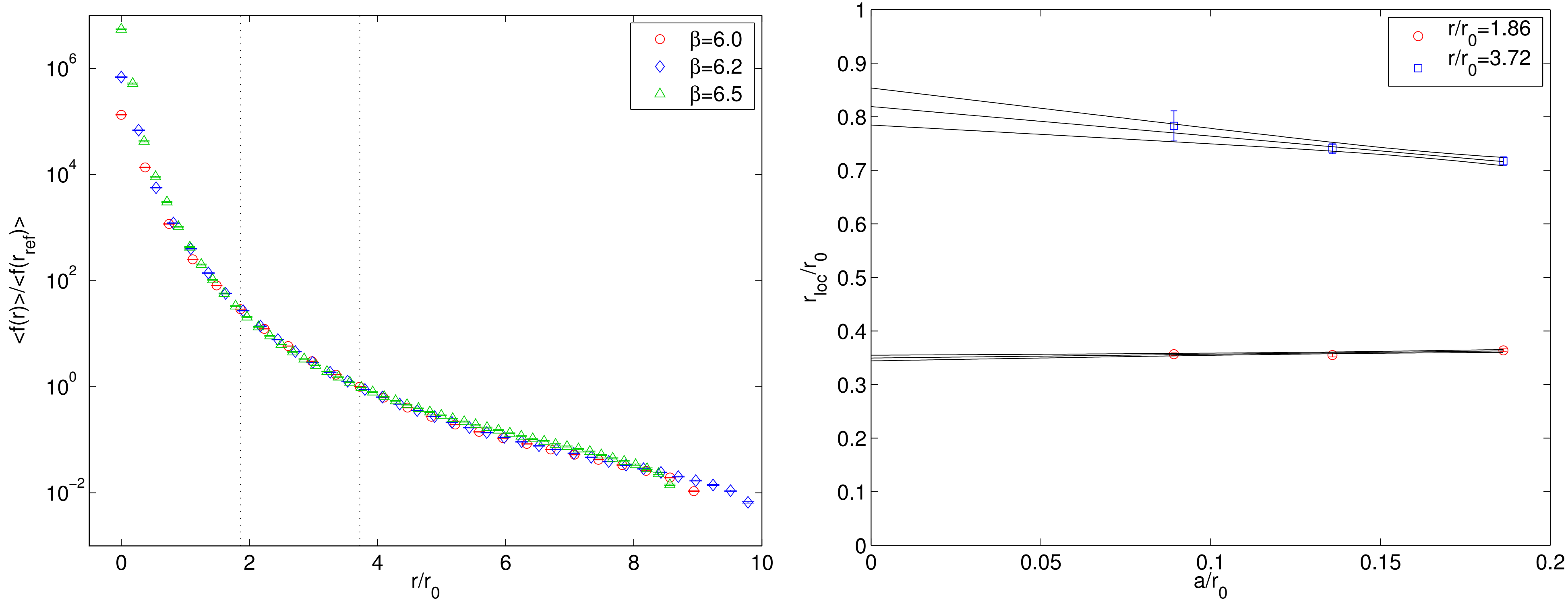
$\Longrightarrow$ $D_{\rm st}^{1/2}$, $D_{\rm st}^{1/4}$ (in 2D, 4D) unacceptable, since $\hat{D} = \mathrm{i}\gamma_\mu p_\mu + O(p^2)$ violated (and non-analytic in p).

- **explicit non-locality of $(D_{\rm st}^\dagger D_{\rm st})^{1/2}_{\rm ee}$**

Measure of localization: $f(r) = \sup\{\ ||\psi(x)||_2 \,\big|\, ||x-y||_1 = r\ \}$ where $\psi(x) = \sum D_{\rm ca}(x,y)\eta(y)$ with η normalized random vector at y.

$\longrightarrow$ For local $D_{\rm ca}$ one has $f(r) \propto e^{-r/r_{\rm loc}}$, ideally with $r_{\rm loc} \propto a$ (in any case $\xi_{\rm phys}/r_{\rm loc} \to \infty$).

B.Bunk, M.DellaMorte, K.Jansen, F.Knechtli, NPB 697, 343 (2004) [hep-lat/0403022]



Rooted operator $D_{\rm ca} = (D_{{\rm st},m}^\dagger D_{{\rm st},m})^{1/2}_{\rm ee} = (-D_{\rm st}^2 + m^2)^{1/2}_{\rm ee} > 0$ as a first test.

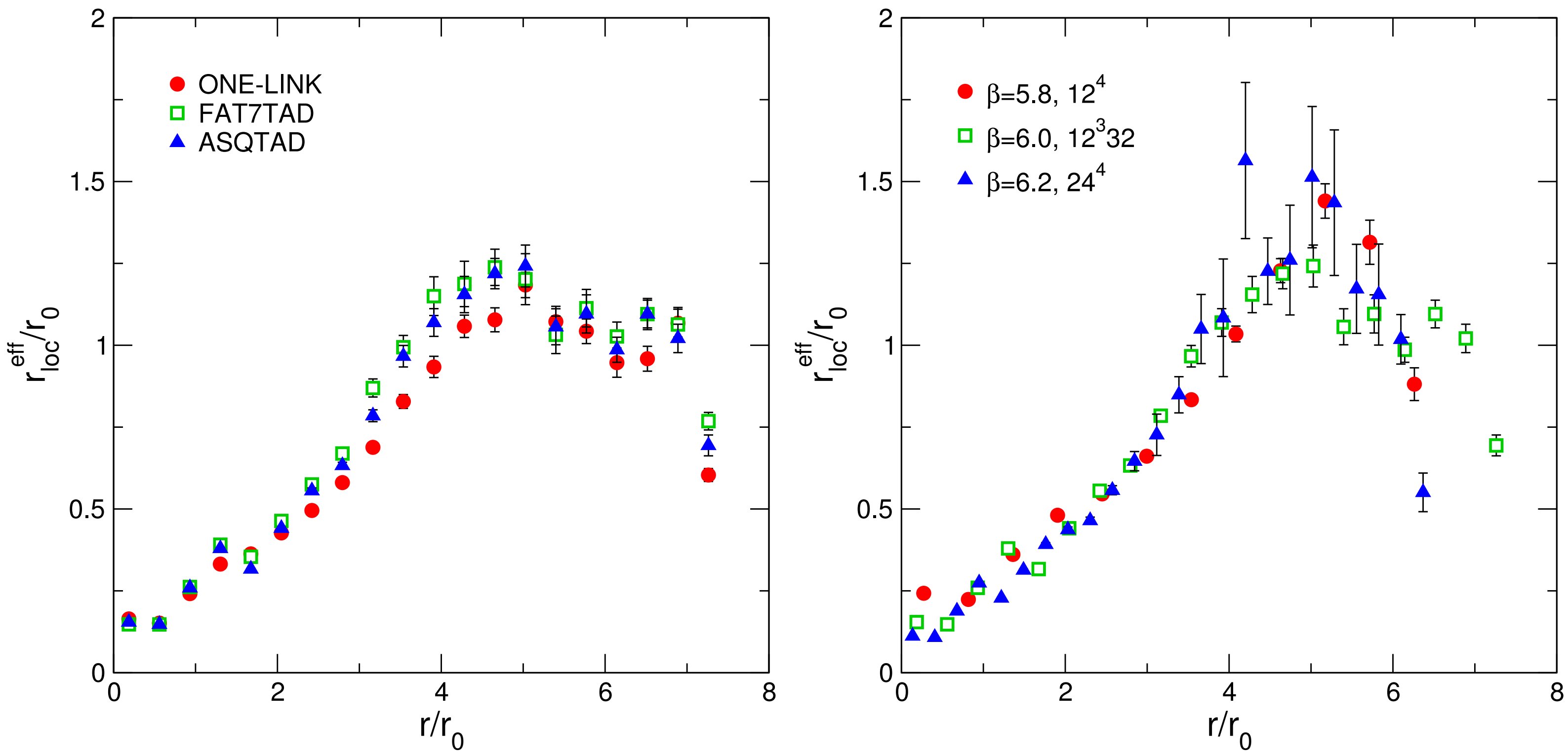
$\longrightarrow$ $f(r)$ and $r_{\rm loc}(r)$ are finite quantities; they scale even though (for a local $D_{\rm ca}$) they should not.

⟶ Bunk et al. show: $r_{\rm loc} = {\rm fcn}(r/r_0)/M_\pi$ in the interacting theory (numerically).

⟶ Bunk et al. show: $r_{\rm loc} = {\rm const}/m$ in the free theory (analytically).

⟶ In physical units $r_{\rm loc}$ constant under $\beta \to \infty$, thus $(D_{\rm st}^\dagger D_{\rm st})^{1/2}_{\rm ee}$ is non-local.

A.Hart, E.Müller, PRD 70, 057502 (2004) [hep-lat/0406030]



⟶ Same conclusion (of course) with improved/filtered staggered quarks.

⟹ Question: Is there one local $D_{\rm ca}$ with $\det(D_{\rm ca}) = {\rm const}\cdot\det^{1/4}(D_{\rm st})$ or modulo cut-off effects ?

Free case: four candidates

In the free case $\mathrm{spec}(D_{\mathrm{st}})$ highly degenerate, thus "thinning" of d.o.f. much easier.

- **D.Adams, hep-lat/0411030**

In the taste basis $D_{\mathrm{st},m} = \nabla_\mu(\gamma_\mu \otimes I_4) - \frac{b}{2}\triangle_\mu(\gamma_5 \otimes \tau_\mu\tau_5) + m(I_4 \otimes I_4)$ on the blocked lattice may be used to build an operator which is simultaneously diagonal in spinor⊗taste

$$D^\dagger_{\mathrm{st},m} D_{\mathrm{st},m} = [-\nabla^2 + \frac{b^2}{4}\triangle^2 + m^2](I_4 \otimes I_4) \; .$$

On the blocked lattice a free generalized Wilson operator $D_{\mathrm{ca},m} = \nabla_\mu\gamma_\mu + \frac{b}{2}W + m$ yields

$$D^\dagger_{\mathrm{ca},m} D_{\mathrm{ca},m} = [-\nabla^2 + (\frac{b}{2}W + m)^2](I_4 \otimes 1) \; .$$

With $\det(D^\dagger_{\mathrm{st},m}) = \det(D_{\mathrm{st},m})$ and $\det(D^\dagger_{\mathrm{W},m}) = \det(D_{\mathrm{W},m})$ it follows that

$$\frac{b^2}{4}\triangle^2 + m^2 = (\frac{b}{2}W + m)^2 \quad \Longrightarrow \quad \det{}^{1/4}(D_{\mathrm{st},m}) = \det(D_{\mathrm{ca},m}) \; .$$

$\Longrightarrow$ $D_{\mathrm{ca},m} = \nabla_\mu\gamma_\mu + \sqrt{\frac{b^2}{4}\triangle^2 + m^2}$ (in the free theory, on the blocked lattice)

$\longrightarrow$ $r_{\mathrm{loc}} = \sqrt{8a/m} \xrightarrow{a\downarrow 0} 0$ (i.e. local, but requires $m > 0$)

- **F.Maresca, M.Peardon, hep-lat/0411029**

On the blocked lattice a free 1-taste operator $D_{\mathrm{ca}} = i\gamma_\mu P_\mu + Q$ with local (i.e. not ultra-local) P_μ, Q yields $\det(D_{\mathrm{ca}}) = \det^{1/4}(D_{\mathrm{st}})$, if $D_{\mathrm{ca}}^\dagger D_{\mathrm{ca}} \otimes I_4^{\mathrm{taste}} = D_{\mathrm{st}}^\dagger D_{\mathrm{st}}$. With $P_\mu = P_\mu^\dagger, Q = Q^\dagger$ trivial in spinor space and $[P_\mu, Q] = 0$ one thus requires that $D_{\mathrm{ca}}^\dagger D_{\mathrm{ca}} = P_\sigma P_\sigma + Q^2 \overset{(!)}{=} [-\triangle + \frac{b^2}{4}\triangle^2] \otimes I_4^{\mathrm{spinor}}$.

Ansatz:

$$P_\mu = \sum_{r\geq 0} \sum_{|d|\leq r} \omega_{p,\mu}^r(x,y)$$

$$Q = \sum_{r\geq 0} \sum_{|d|\leq r} \omega_q^r(x,y)$$

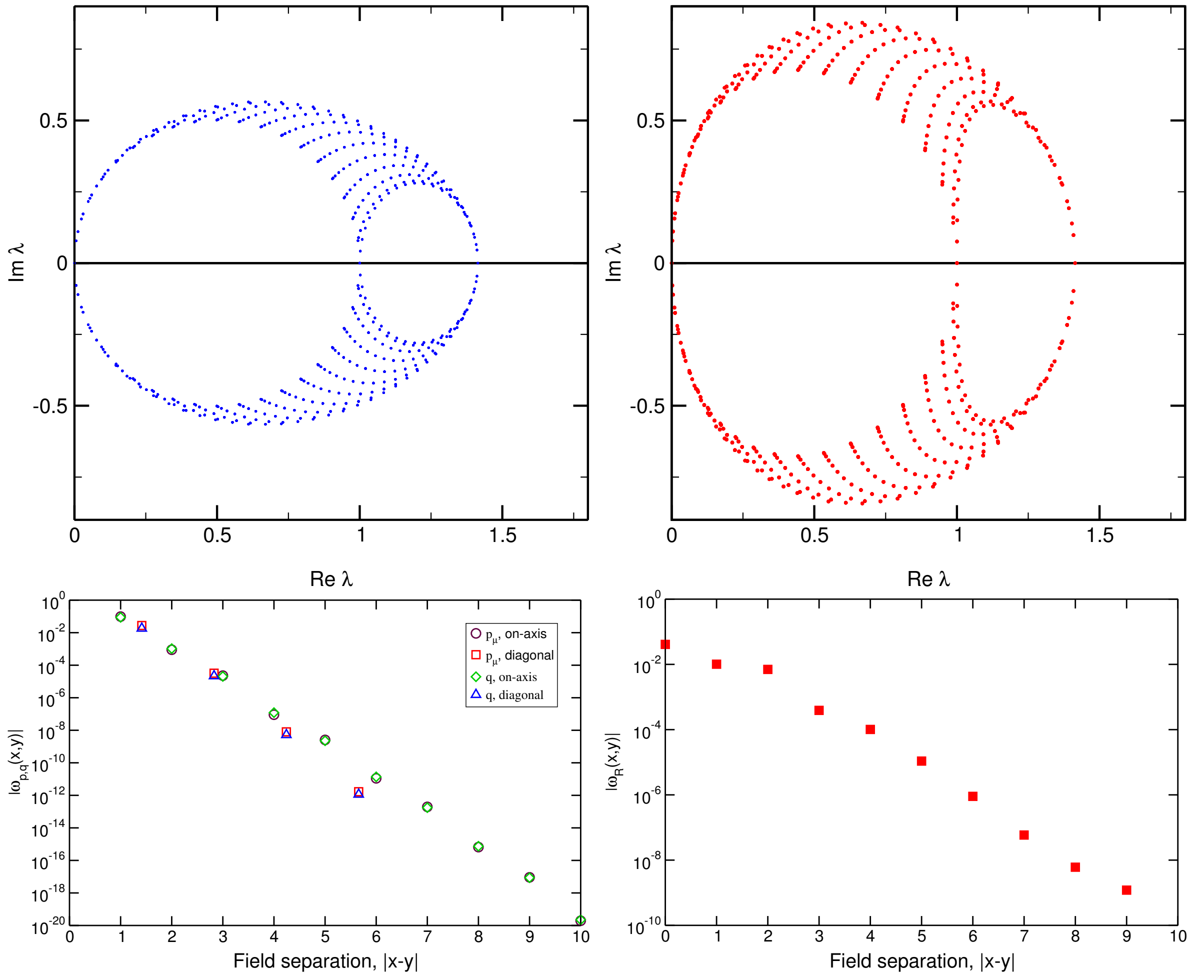
where $\omega_{p,\mu}^r, \omega_q^r$ have range r.

Solution 1:
Without further constraints, optimizing locality of D_{ca} yields spectrum and fall off pattern of $\omega_{p,\mu}^r, \omega_q^r$ shown on the left.

Solution 2:
Ditto, but restriction to $Q = -\triangle R$ with local R yields spectrum and (slower) fall off pattern shown on the right and

$$\{D_{\mathrm{ca}}, \gamma_5\} = D_{\mathrm{ca}}\, 2R\gamma_5\, D_{\mathrm{ca}}.$$

[Sol. 1 for $m > 0$ even better localized.]



- Y.Shamir, PRD 71, 034509 (2005), hep-lat/0412014

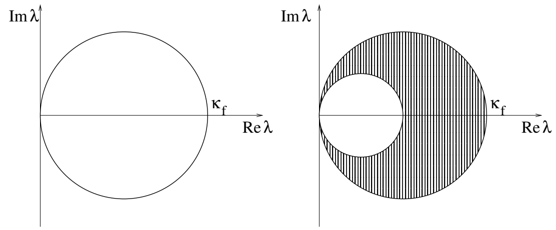
Main idea: Improve taste symmetry through RG blocking. Infinitely many blocking steps would achieve $D_n \to D_\infty \otimes I_4$, while $D_{\rm ca}$ after n steps satisfies $\det^{1/4}(D_{\rm st}) = \det(D_{\rm ca})\det^{1/4}(T)$ where T contains only cut-off excitations and should maintain Symanzik class, i.e. $\det(T) = \mathrm{const}\cdot(1+O(a^2))$.

Note: If one is satisfied with $a=0.4\,\mathrm{fm}$ for $D_{\rm ca}$ (optimistic view), then original lattice with $a\sim 0.1\,\mathrm{fm}$ allows for 2 steps; for 5 steps original lattice must have $a\sim 0.01\,\mathrm{fm}$ (cf. talk by F. Maresca).

The massless staggered action on the original lattice satisfies $\{D_0,(\gamma_5\otimes\tau_5)\} = 0$ or equivalently $(\gamma_5\otimes\tau_5)D_0(\gamma_5\otimes\tau_5) = D_0^\dagger$. After $n\geq 1$ RG steps (parameter α_n) one has the generalized GW relation

$$\{D_n^{-1},(\gamma_5\otimes\tau_5)\} = \mathrm{const}\cdot\delta_{x,y} \qquad \text{or} \qquad \{D_n,(\gamma_5\otimes\tau_5)\} = D_n\frac{2}{\alpha_n}(\gamma_5\otimes\tau_5)\,D_n\,.$$

If one could establish $(\gamma_5\otimes\tau_5)$-hermiticity of D_n, one would easily obtain $D_n + D_n^\dagger = D_n\frac{2}{\alpha_n}D_n^\dagger$, i.e. spectrum on a circle.



With $D_n = \sum_j \lambda_j u_j v_j^\dagger$ and $v_i^\dagger u_k = \delta_{i,k}$, upon sandwiching $v_i^\dagger(\mathrm{GW})u_k$, one obtains $\lambda_i v_i^\dagger(.\otimes.)u_k + v_i^\dagger(.\otimes.)u_k\lambda_k = \lambda_i v_i^\dagger\frac{2}{\alpha_n}(.\otimes.)u_k\lambda_k$ and thus for arbitrary pair (i,k) of L,R-eigenmodes $v_i^\dagger, u_k$ that $v_i^\dagger(\gamma_5\otimes\tau_5)u_k = 0$ or $\lambda_i+\lambda_k-\frac{2}{\alpha_n}\lambda_i\lambda_k = 0$. In particular for $i=k$ it follows that $v_j^\dagger(\gamma_5\otimes\tau_5)u_j \neq 0$ implies $2\lambda_j - \frac{2}{\alpha_n}\lambda_j^2 = 0$ or $\lambda_j\in\{0,\alpha_n\}$.

- J.Giedt, hep-lat/0507002 Similar concepts – exploratory discussion of interacting case.

- H.Neuberger, PRD 70, 097504 (2004), hep-lat/0409144 Issue cast into local field-theoretical framework in 6D.

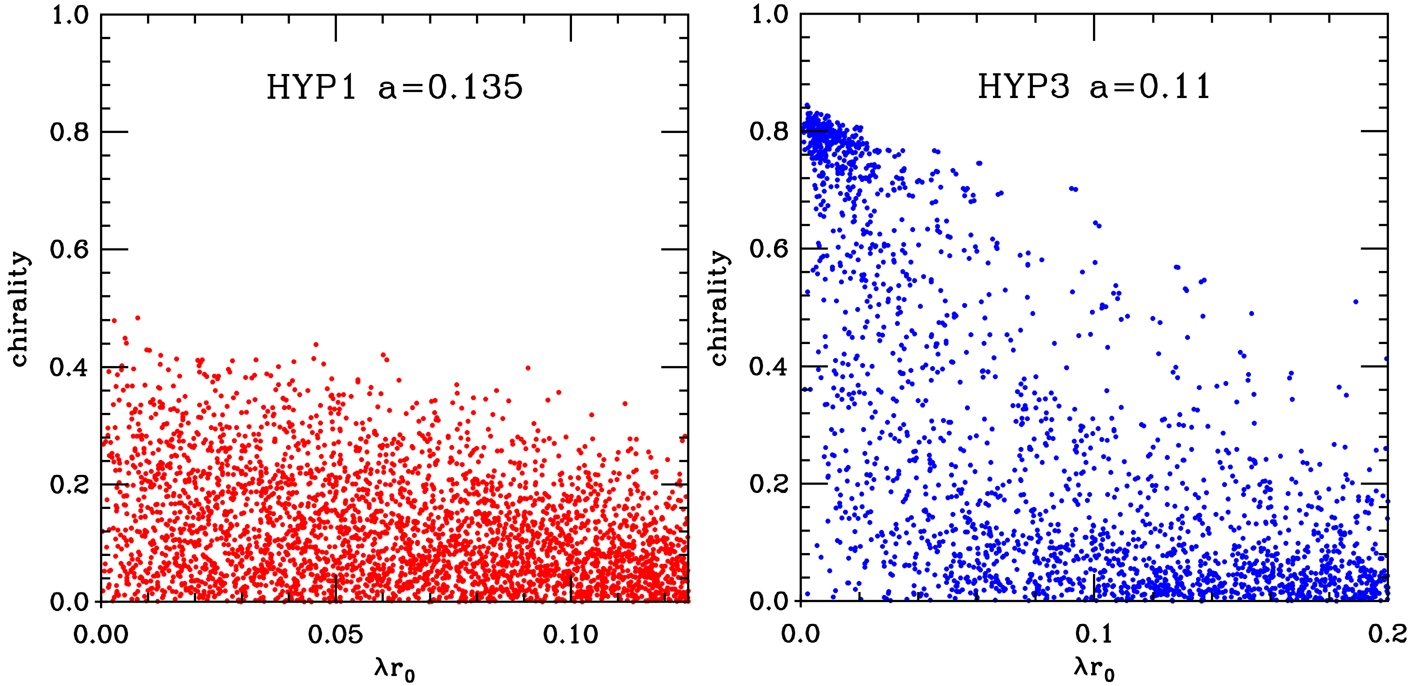
Interacting theory: $\mathrm{spec}(D_{\mathrm{st}})$ in 4D

- **concept of filtering**

— Replace covariant derivative, e.g. $U_\mu(x)\psi(x+\hat{\mu}) - \psi(x) \;\rightarrow\; U_\mu^{\mathrm{HYP}}(x)\psi(x+\hat{\mu}) - \psi(x)$.
— Redefines (diminishes) cut-off effects without changing Symanzik class, i.e. $O(a^2) \rightarrow O(a^2)$.
— Designed to render D_{st} “immune” against $p \sim \pi/a$ gluons, impact on taste “symmetry” ?

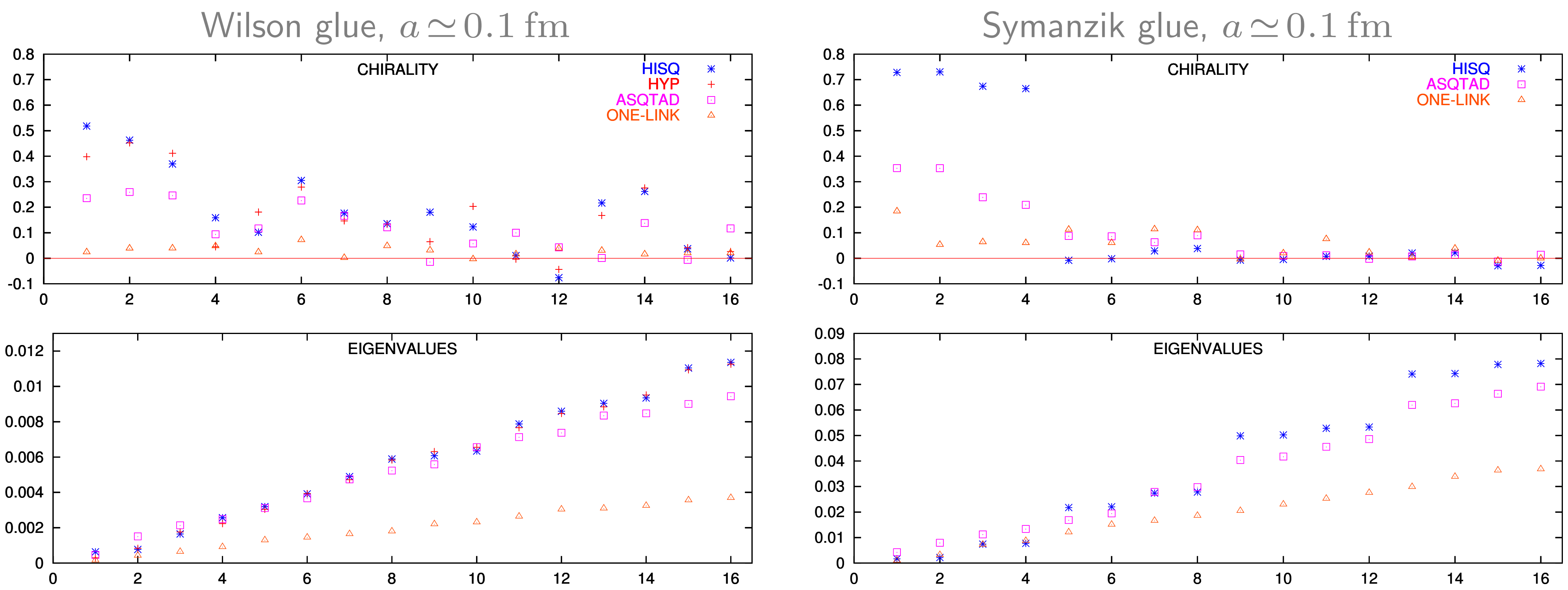
- **chirality of low-lying eigenmodes**

Continuum: $D\psi = \lambda\psi \iff D\gamma_5\psi = -\lambda\gamma_5\psi$ and $\psi^\dagger\gamma_5\psi = 0$ (for $\lambda \neq 0$).
$D\zeta = 0$ and $\zeta^\dagger\gamma_5\zeta = \pm 1$ characteristic signature of zero-modes.



A.Hasenfratz, http://www.rccp.tsukuba.ac.jp/lat03/Dat/OHP/a.hasenfratz.ps

- **near-degeneracy of staggered quartets**



E.Follana, A.Hart, C.T.H.Davies, PRL 93, 241601 (2004) [hep-lat/0406010]

E.Follana, NPPS 140, 141 (2005) [hep-lat/0409062]

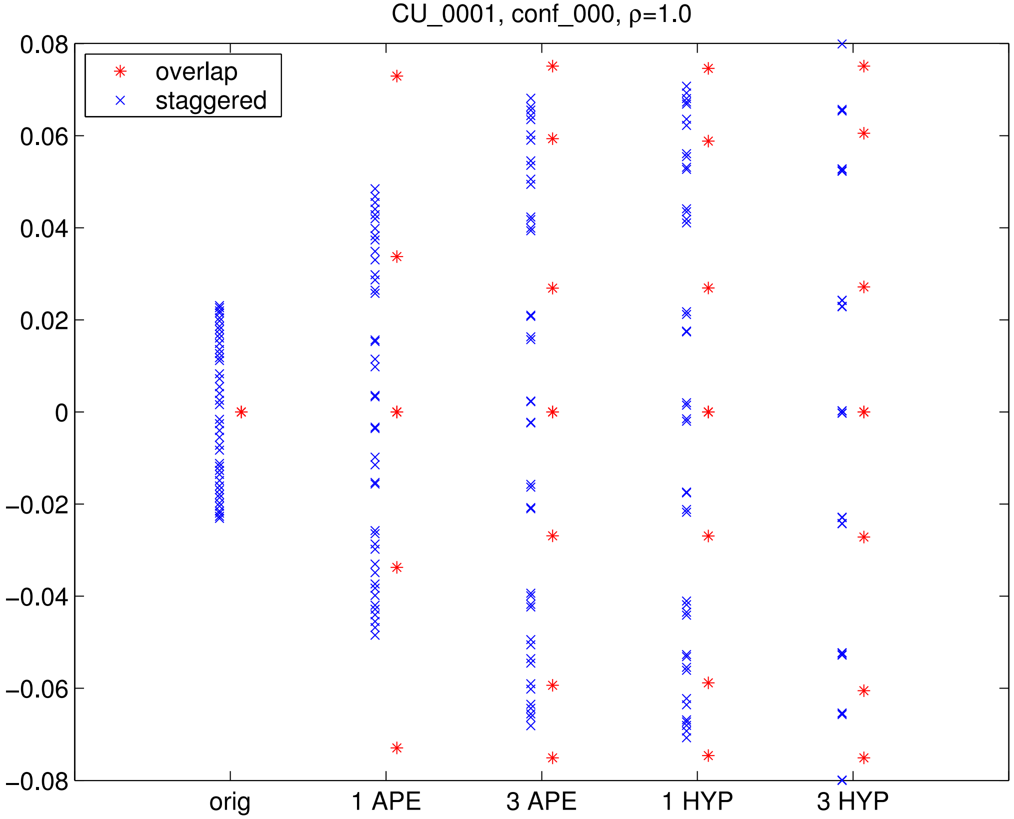
⟶ Fine lattice plus filtering makes separation into near-zero modes and non-zero modes visible.

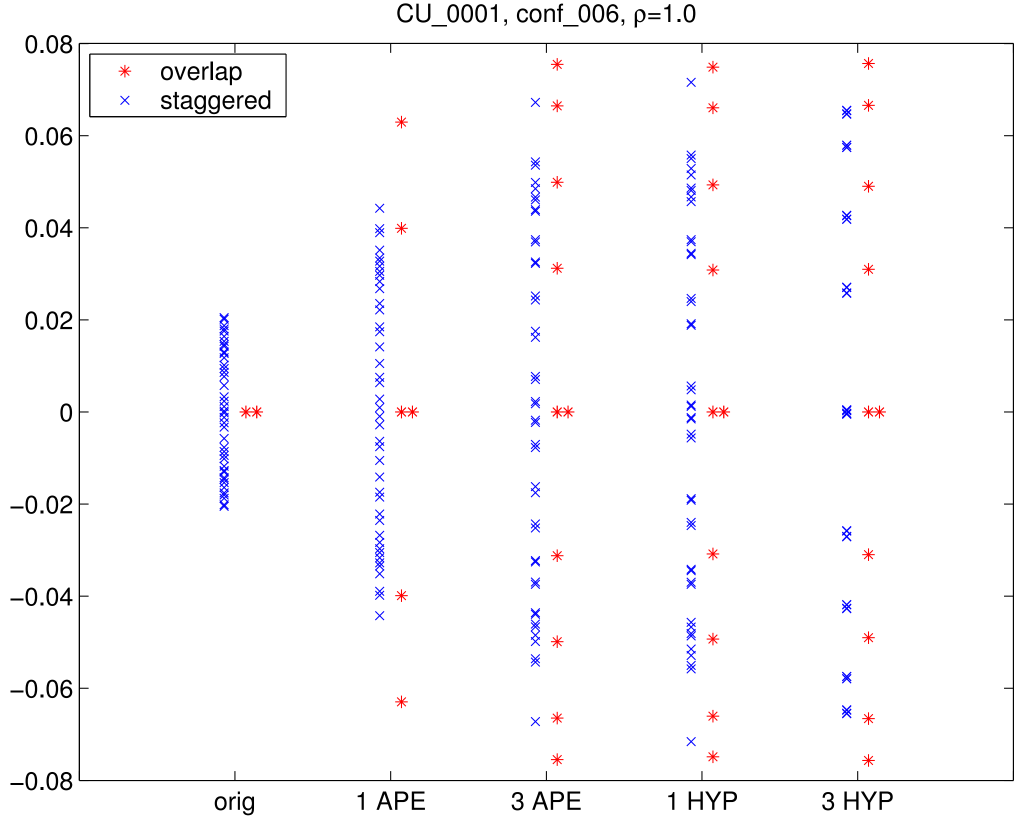
⟶ Near-zero modes have chirality $|\zeta^\dagger\gamma_5\zeta| \simeq 1$, non-zero modes have $\psi^\dagger\gamma_5\psi \simeq 0$.

⟹ Approximate index theorem for staggered fermions (once taste "symmetry" visible).

• comparing with overlap spectrum on individual configurations

CU_0001: staggered $N_t = 2$ simulation ($\beta = 5.7, m = 0.01 \to a \simeq 0.1$ fm)





SD, C.Hoelbling, U.Wenger, PRD 70, 094502 (2004) [hep-lat/0406027]

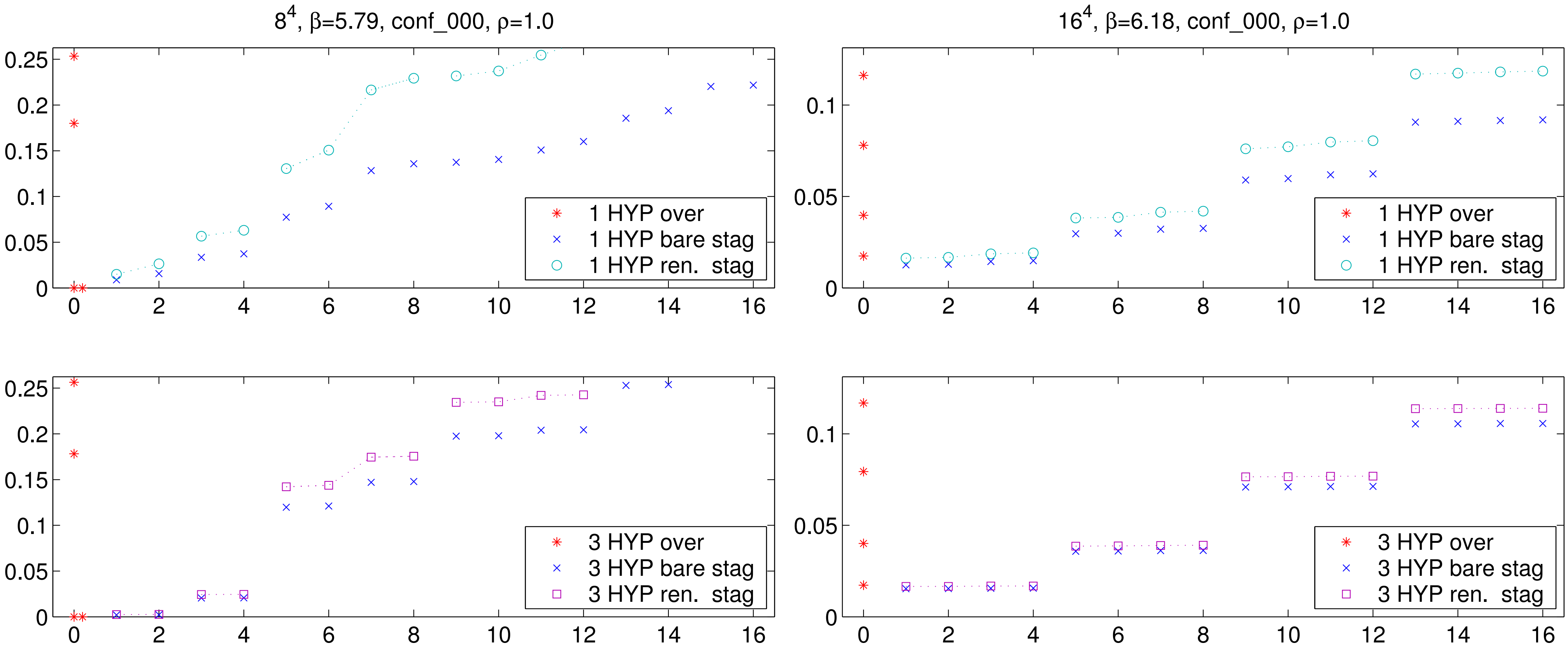
$\longrightarrow$ Filtering pushes $\lambda_{\rm st}$ out and pulls $\hat{\lambda}_{\rm ov}$ in (in general: drives $Z_S \to 1$).

$\longrightarrow$ Manifest staggered quartet to single overlap mode correspondence (modulo different Z_S factor).

$\Longrightarrow$ In particular: $4|q|$ staggered near-zero modes on (typical) configuration with overlap charge q.

• cut-off dependence of taste-splitting

Matched lattices ($\beta = 5.66, 5.79, 6.00, 6.18$, $L^4 = 6^4, 8^4, 12^4, 16^4$) with $V = (1.12\,\mathrm{fm})^4 = 1.57\,\mathrm{fm}^4$.



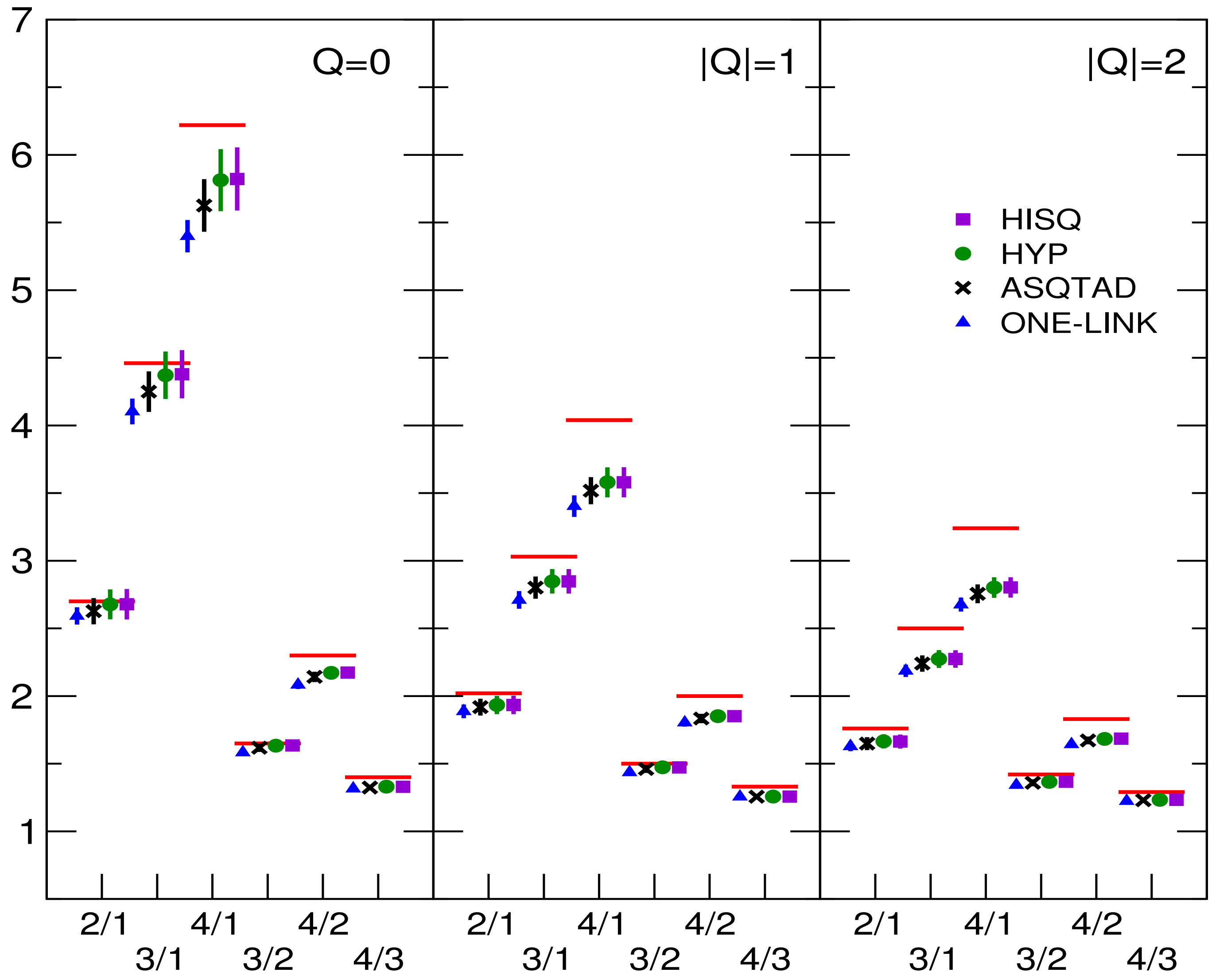
SD, C.Hoelbling, U.Wenger, PRD 70, 094502 (2004) [hep-lat/0406027]

$\longrightarrow$ On matched lattices taste "symmetry" (quartet near-degeneracy) improves with $a \to 0$.

$\longrightarrow$ Rescaling with $Z_S^{\mathrm{ov}}/Z_S^{\mathrm{st}}$: quantitative agreement of staggered quartet with single overlap mode.

$\Longrightarrow$ Explicit tests passed by D_{ov} (index theorem, RMT agreement, Banks-Casher, ...) transfer to D_{st}.

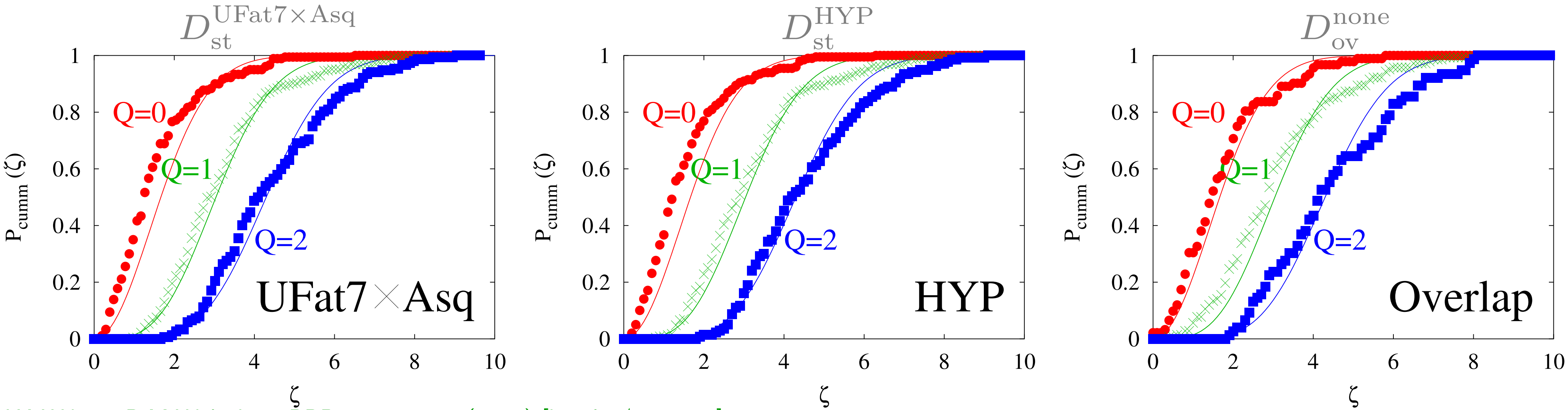
- **explicit check that $\langle\lambda_i\rangle/\langle\lambda_j\rangle$ agrees with RMT prediction**



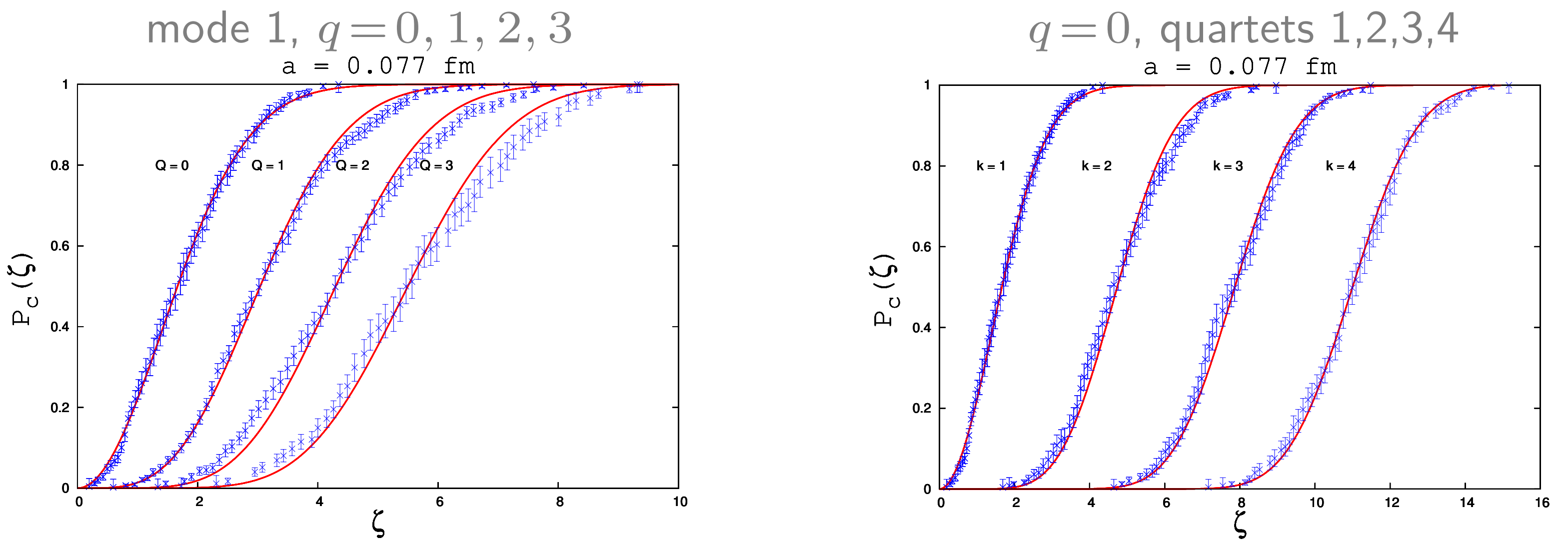
E.Follana, A.Hart, C.T.H.Davies, PRL 93, 241601 (2004) [hep-lat/0406010]

$\longrightarrow$ Sectoral $\langle\lambda_i\rangle/\langle\lambda_j\rangle$ agrees with RMT prediction (up to small finite volume effects ?).

- **explicit check that CED($\lambda_{\min}$) agrees with RMT prediction**



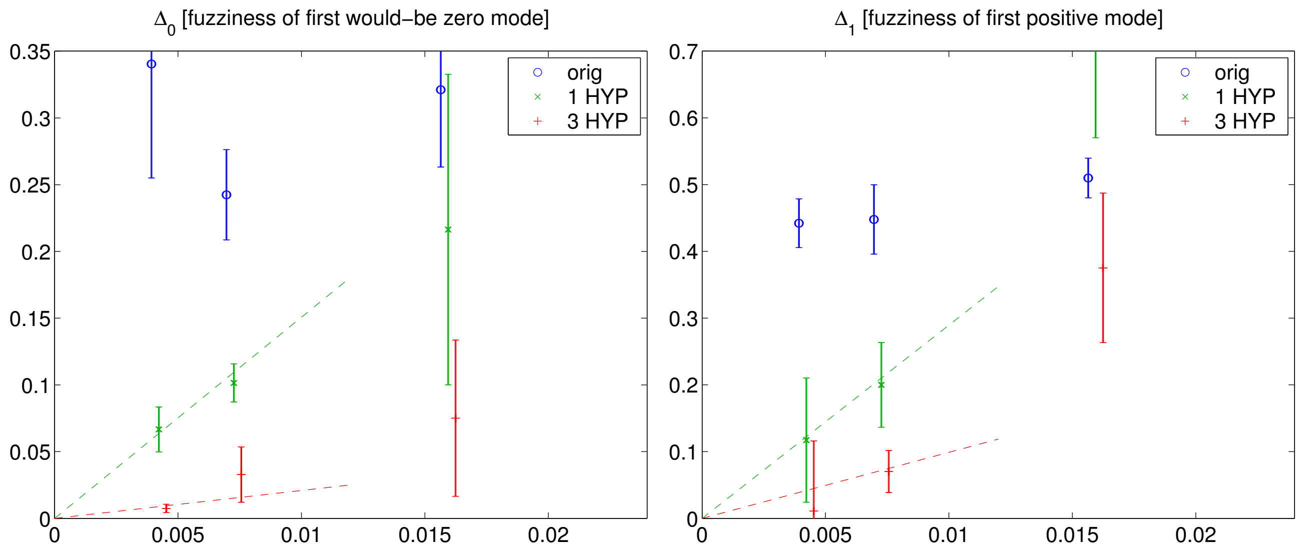
K.Y.Wong, R.M.Woloshyn, PRD 71, 094508 (2005) [hep-lat/0412001]



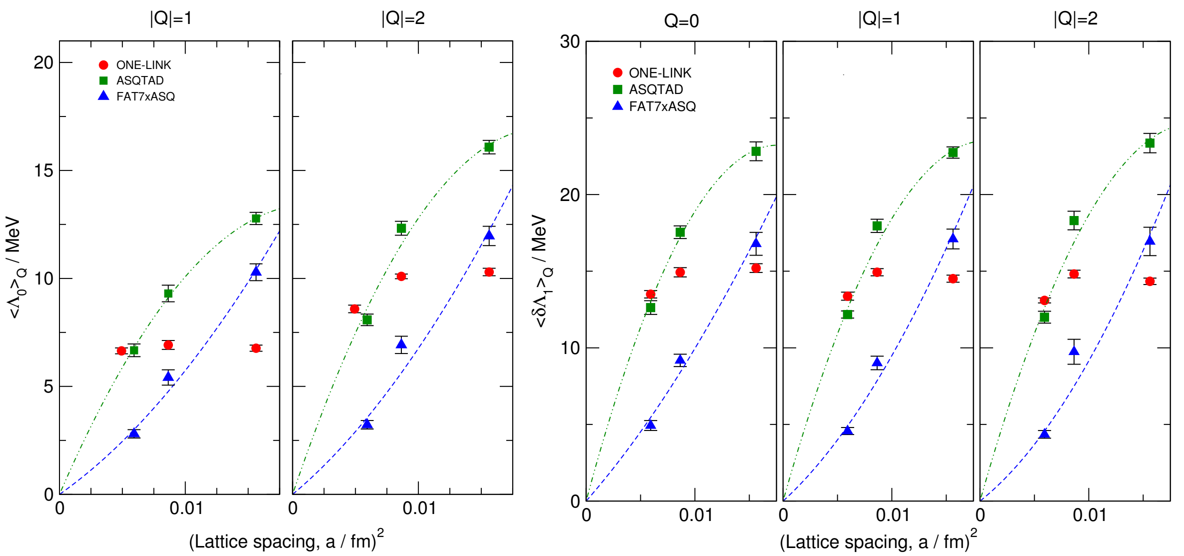
E.Follana, A.Hart, C.T.H.Davies, Q.Mason, hep-lat/0507011

$\Longrightarrow$ On fine enough lattices (and with filtering) agreement with RMT for individual topological sectors.

- **evidence that taste-splitting is $O(a^2)$ effect**



SD, C.Hoelbling, U.Wenger, PRD 70, 094502 (2004) [hep-lat/0406027]



E.Follana, A.Hart, C.T.H.Davies, Q.Mason, hep-lat/0507011

$\Longrightarrow$ Minimal a for taste breaking to possibly be $O(a^2)$ effect seems enlarged through filtering.

Interacting theory: $\chi_{\rm sca}\,,\chi_{\rm top}$ in 2D

- In 2D rooting issue exists for $N_f=1,3,\ldots$, since $D_{\rm st}$ yields 2 fermions in the continuum.
- In 2D scale may be set through fundamental coupling: $[g]=[e]=1$ (no UV running), $\beta=1/(ag)^2$.
- Analytic $N_f=1$ result: $\lim_{m=0}\langle\bar\psi\psi\rangle/g = e^\gamma/(2\pi^{3/2}) = 0.1599\ldots$ (anomaly induced J.Schwinger (1962))

overlap fermions ($\rho=1$):

$$\frac{\chi^{\rm ov}_{\rm sca}}{g} = \frac{\sqrt{\beta}}{V}\,\frac{\langle \det^{N_f}(D^{\rm ov}_m)\sum\frac{1}{\hat\lambda+m}\rangle}{\langle \det^{N_f}(D^{\rm ov}_m)\rangle} \quad \text{(bare)}$$

$$\det(D^{\rm ov}_m) = \prod((1-\tfrac{m}{2})\lambda+m)$$

$$\hat\lambda = \frac{1}{(1/\lambda-1/2)} \quad \text{(=stereogr. proj.)}$$

staggered fermions:

$$\frac{\chi^{\rm st}_{\rm sca}}{g} = \frac{\sqrt{\beta}}{2V}\,\frac{\langle \det^{N_f/2}(D^{\rm st}_m)\sum\frac{1}{\lambda+m}\rangle}{\langle \det^{N_f/2}(D^{\rm st}_m)\rangle} \quad \text{(bare)}$$

$$\det(D^{\rm st}_m) = \prod(\lambda+m)$$

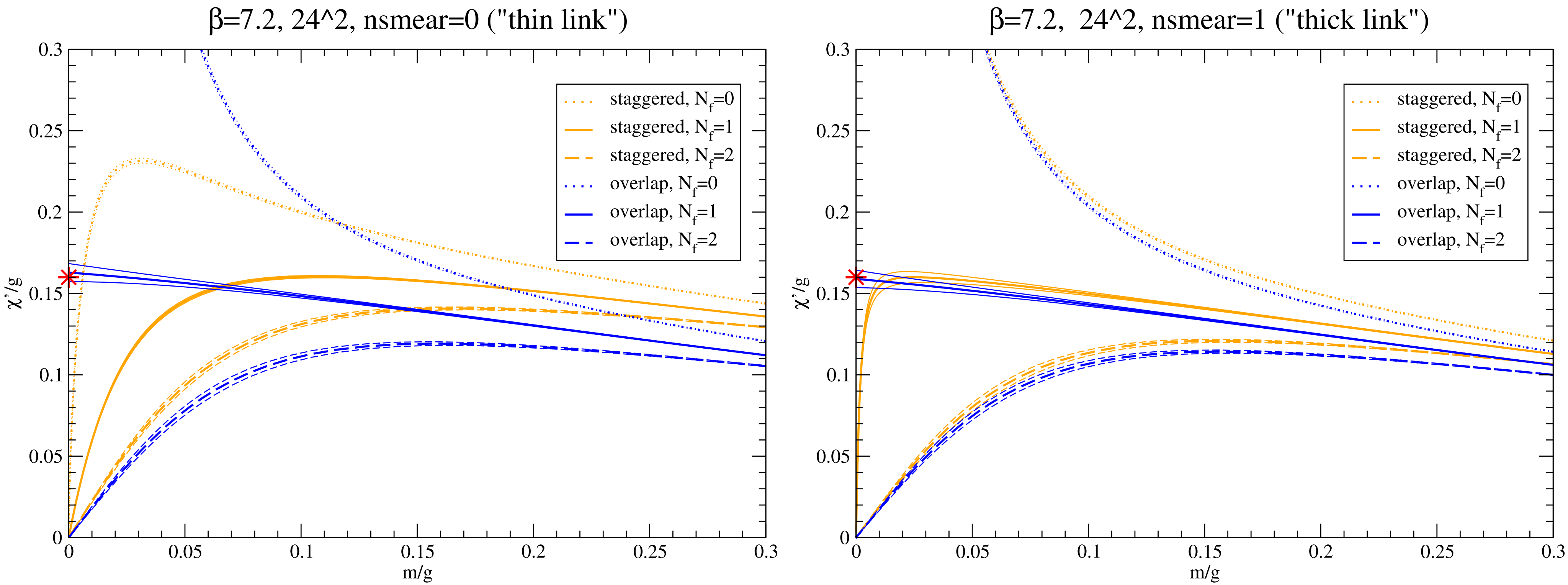
– Sample quenched, compute all λ [LAPACK] and build observables (variable m) in analysis program.

– In plots below theory is unitary (at least with $D_{\rm ov}$); we use $m_{\rm sea}=m_{\rm val}$ throughout.

– Details in SD, C.Hoelbling, PRD 69, 034503 (2004) [hep-lat/0311002]& PRD 71, 054501 (2005) [hep-lat/0411022].

• $\chi_{\rm sca}$ with $D_{\rm st}$ and $D_{\rm ov}$ at $\beta = 7.2$

Continuum: $\chi_{\rm sca}/g \propto \begin{cases} g/m & (N_f = 0) \\ \text{const} & (N_f = 1) \\ m/g & (N_f = 2) \end{cases}$ (in 2D not an order parameter of SSB)



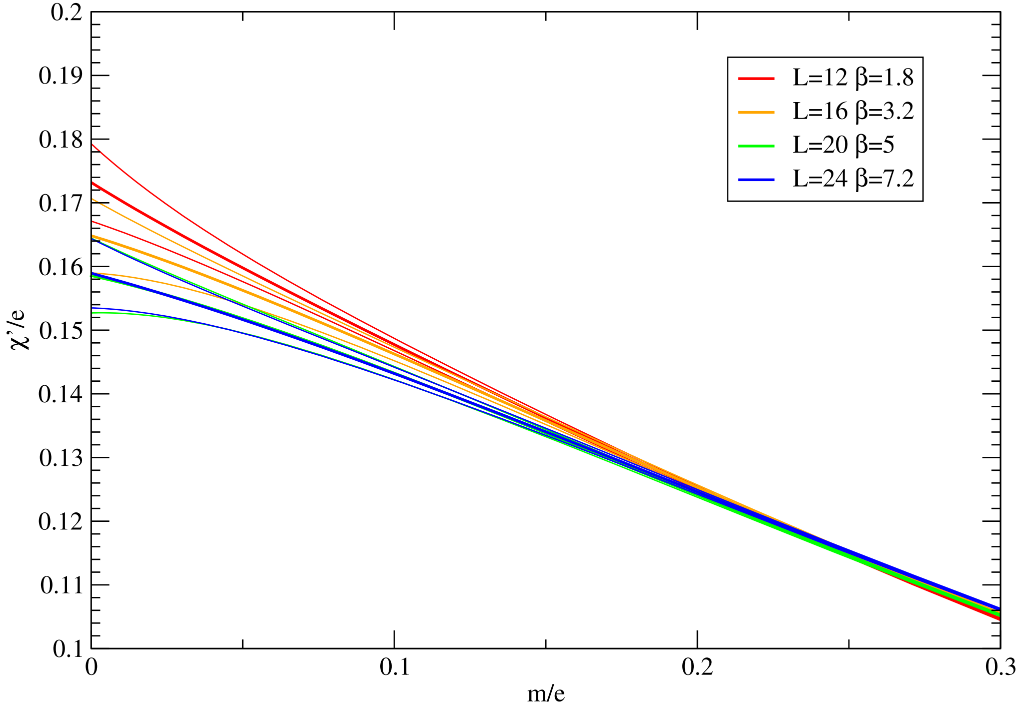
Overlap: qualitatively correct behavior $\forall N_f$, and $\lim_{m\to 0} \chi_{\rm sca}^{N_f=1}/g$ consistent with $0.1599\ldots$ for $\beta \geq 4$.

Staggered: qualitatively wrong behavior in chiral limit for $N_f = 0, 1$, since $\lim_{m\to 0} \chi_{\rm sca}/g = 0$ for any β, but filtering shifts point where staggered answer fails more chiral (rel. to taste splitting ?).

Question: can one obtain $\chi_{\rm sca}/e = 0.1599\ldots$ with staggered fermions, if one first extrapolates to the continuum, taking $m \longrightarrow 0$ afterwards ?

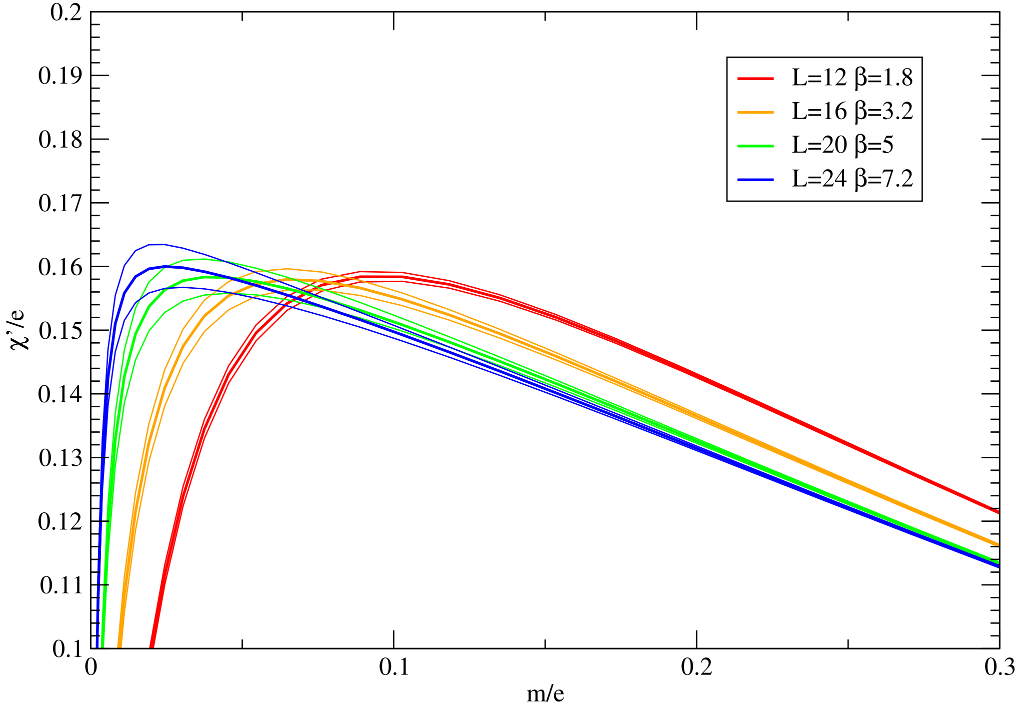
- **$\chi_{\rm sca}$ with $D_{\rm st}$ and $D_{\rm ov}$ in the continuum**

overlap (nsmear=1)

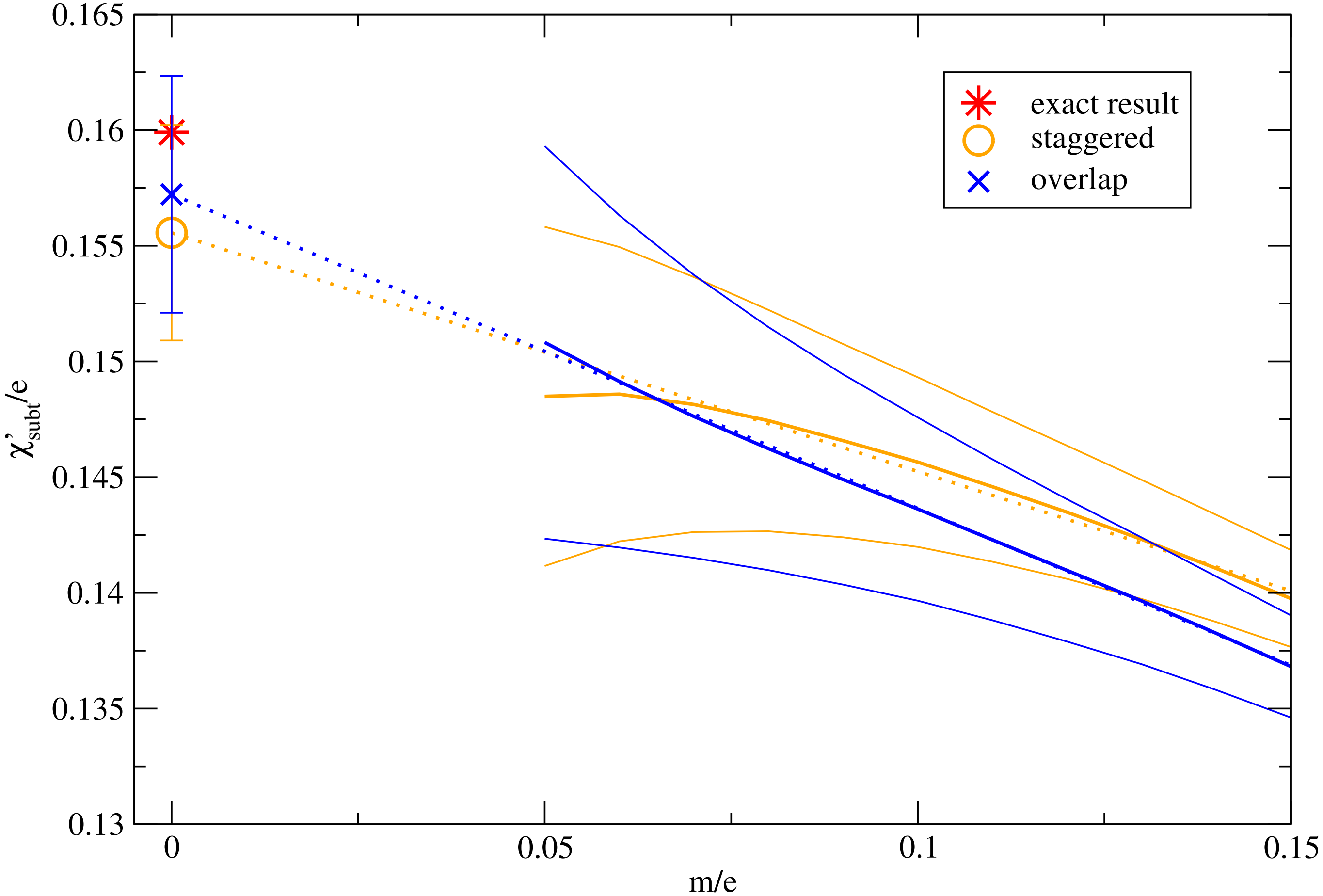


- small cut-off effects
- for $m/e \leq 0.3$ already $\beta \geq 1.8$ sufficient to enter scaling window

staggered (nsmear=1)



- large cut-off effects
- for small m/e ever larger β needed to enter scaling window



- overlap ("universal behavior"):

$$\lim_{a\to0}\lim_{m\to0}\frac{\chi'^{\,\mathrm{ov}}_{\mathrm{sca}}(m/e,a^2)}{e}=\frac{e^\gamma}{2\pi^{3/2}}$$

$$\lim_{m\to0}\lim_{a\to0}\frac{\chi'^{\,\mathrm{ov}}_{\mathrm{sca}}(m/e,a^2)}{e}=\frac{e^\gamma}{2\pi^{3/2}}$$

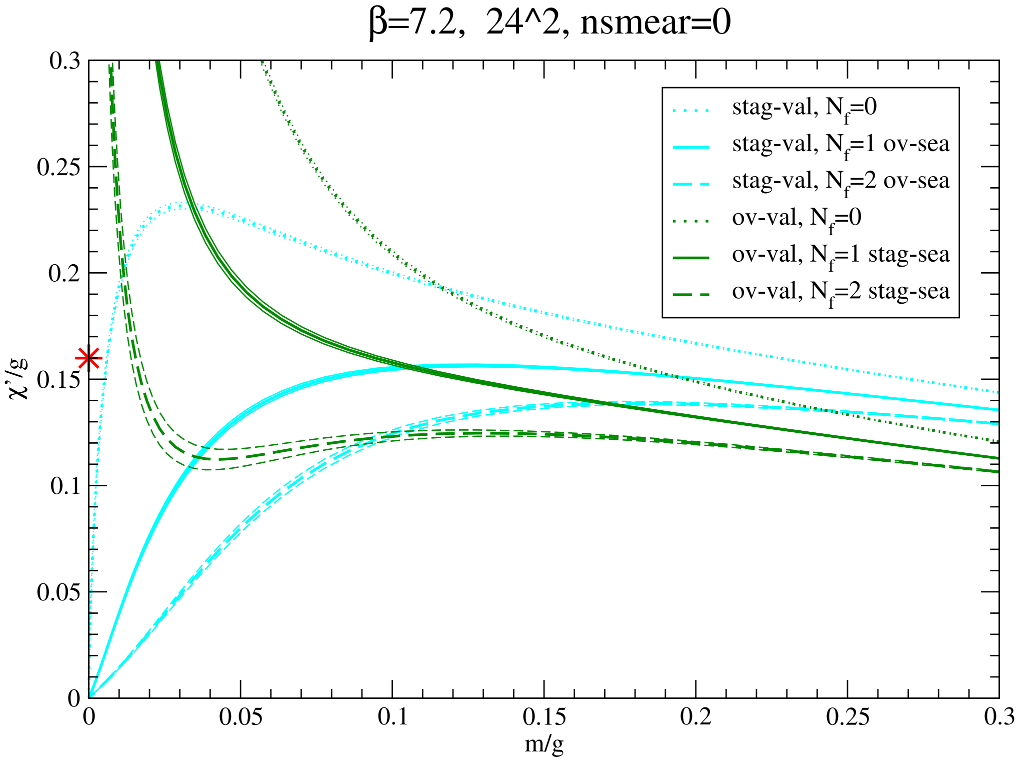
- staggered ("non-commutativity"):

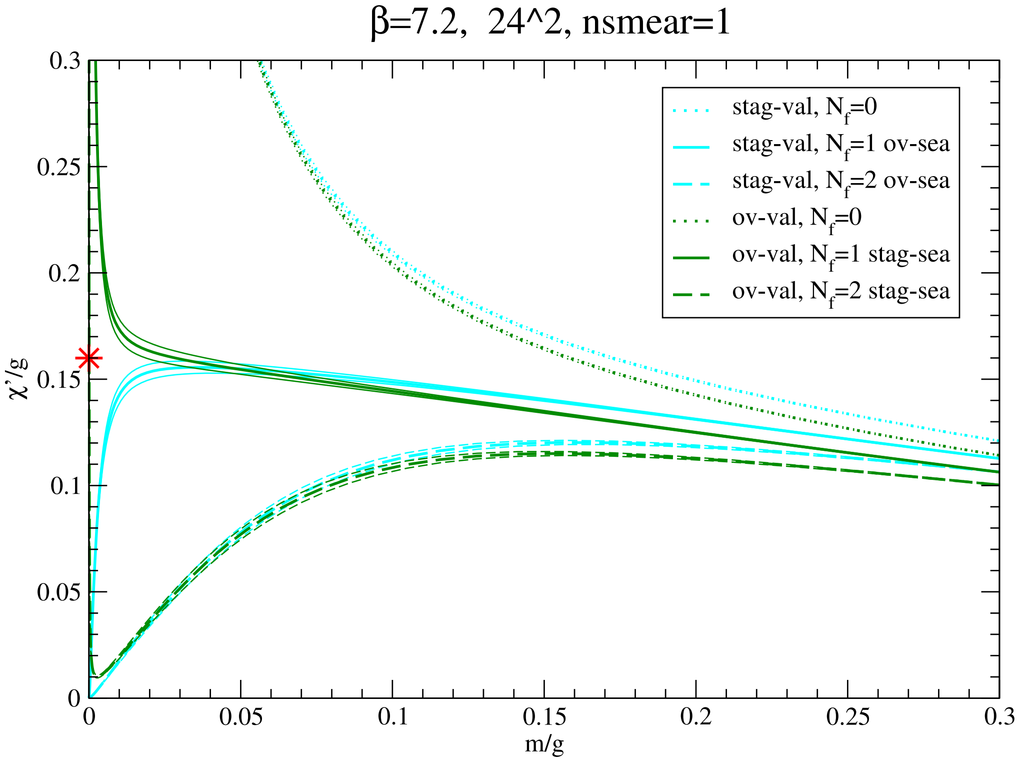
$$\lim_{a\to0}\lim_{m\to0}\frac{\chi'^{\,\mathrm{st}}_{\mathrm{sca}}(m/e,a^2)}{e}=0$$

$$\lim_{m\to0}\lim_{a\to0}\frac{\chi'^{\,\mathrm{st}}_{\mathrm{sca}}(m/e,a^2)}{e}=\frac{e^\gamma}{2\pi^{3/2}}$$

$\Longrightarrow$ Staggered fermions see chiral anomaly, but only if $\lim_{a\to0}$ is taken first and $\lim_{m\to0}$ thereafter.

- χ_{sca} **with D_{st} and D_{ov} in hybrid mode**





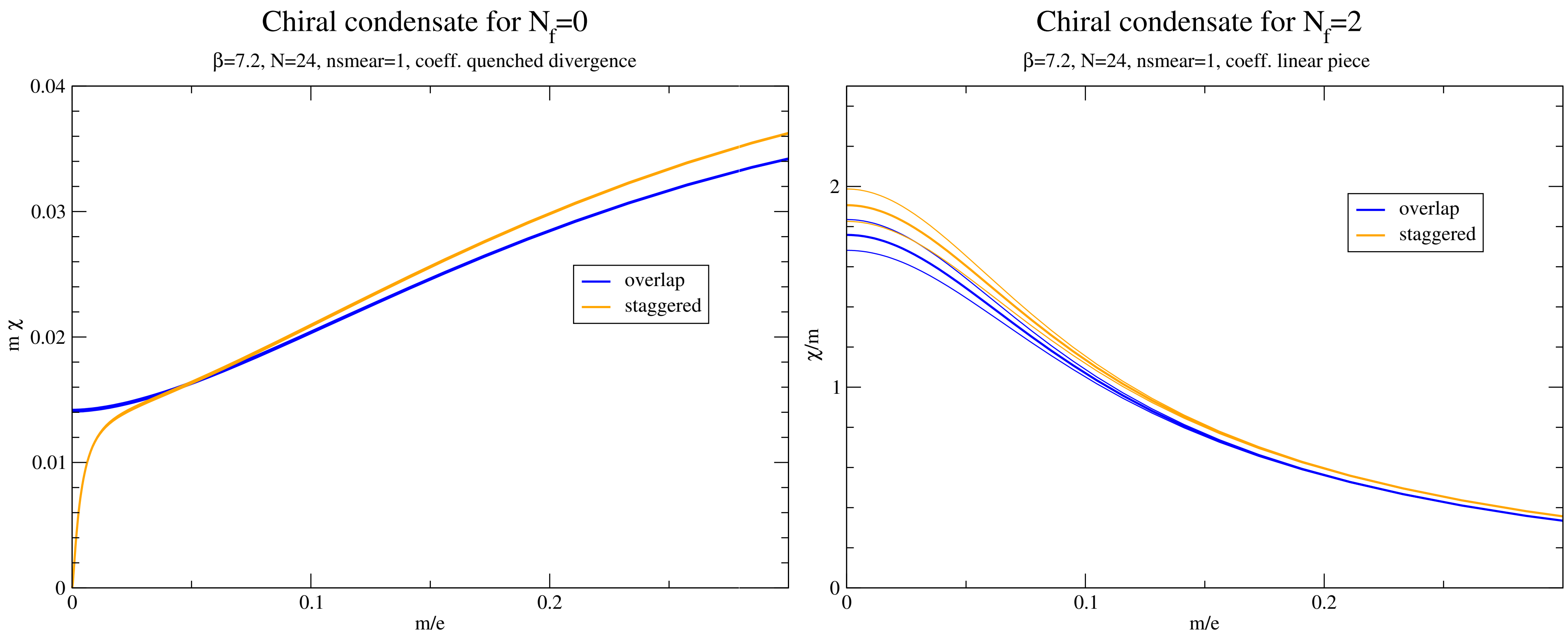
⟶ Warning for all "hybrid action" studies:

Deficiencies of D_{st} in sea/valence sector may overwhelm good properties of D_{ov} in other sector.

⟹ Failure of χ_{scal} in staggered $N_t = 1$ case cannot be attributed to sea or valence sector alone.

- $\chi_{\rm sca}$ **with $D_{\rm st}$ and $D_{\rm ov}$ in the quenched case**

Reminder: $$\chi_{\rm sca}/e \propto \begin{cases} e/m & (N_f=0) \\ m/e & (N_f=2) \end{cases}$$

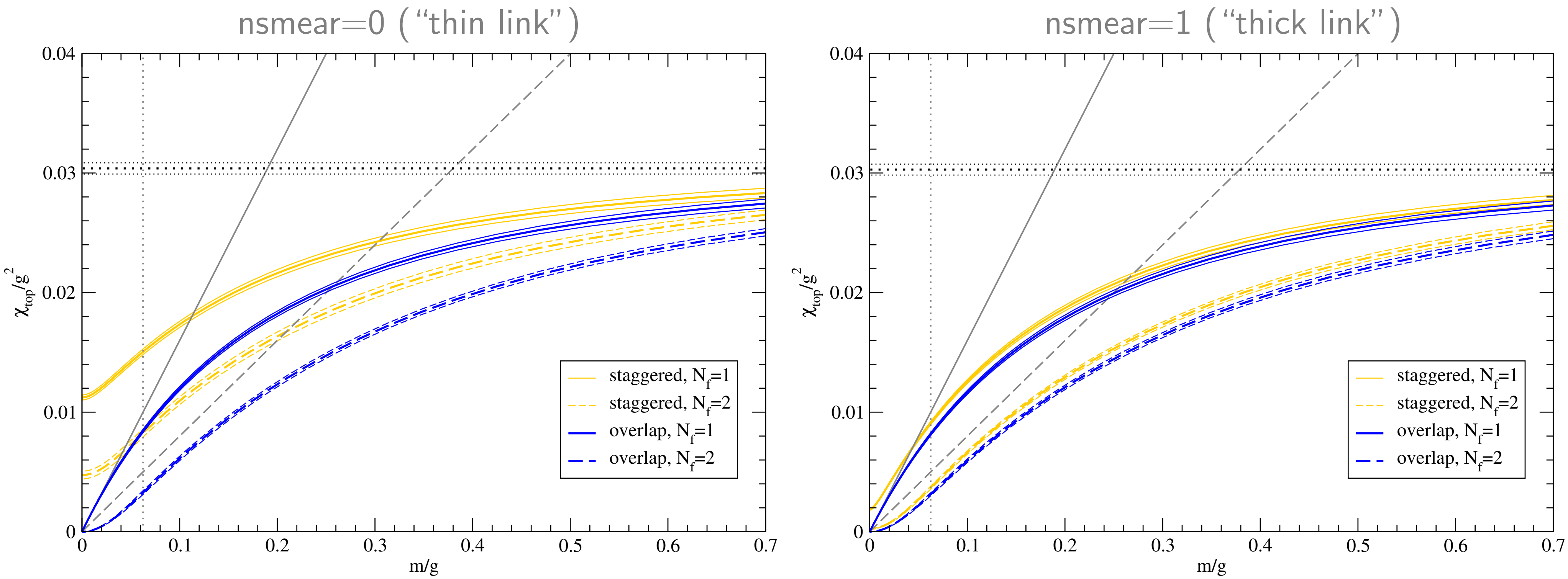


$\longrightarrow$ Even in the quenched theory a staggered non-commutativity (in an artificial observable) found. Earlier paper: J.Smit, J.C.Vink, NPB 286, 485 (1987).

$\Longrightarrow$ Non-commutativity not genuinely tied to rooted determinant, more likely due to mismatch in sea and valence sector; compare discussion in C.Bernard, PRD 71, 094020 (2005) [hep-lat/0412030].

• $\chi_{\rm top}$ with $D_{\rm st}$ and $D_{\rm ov}$ at $\beta=4$

$$\chi_{\rm top}^{\rm ov}=\frac{\beta}{V}\frac{\langle \det^{N_f}(D_m^{\rm ov})q^2\rangle}{\langle \det^{N_f}(D_m^{\rm ov})\rangle} \qquad \chi_{\rm top}^{\rm st}=\frac{\beta}{V}\frac{\langle \det^{N_f/2}(D_m^{\rm st})q^2\rangle}{\langle \det^{N_f/2}(D_m^{\rm st})\rangle} \qquad q=\begin{cases}\mathrm{ind}=-\frac{1}{2}\mathrm{tr}(\gamma_5 D_{\rm ov})\\ \frac{1}{2\pi}\int F_{12}\,d^2x\end{cases}$$

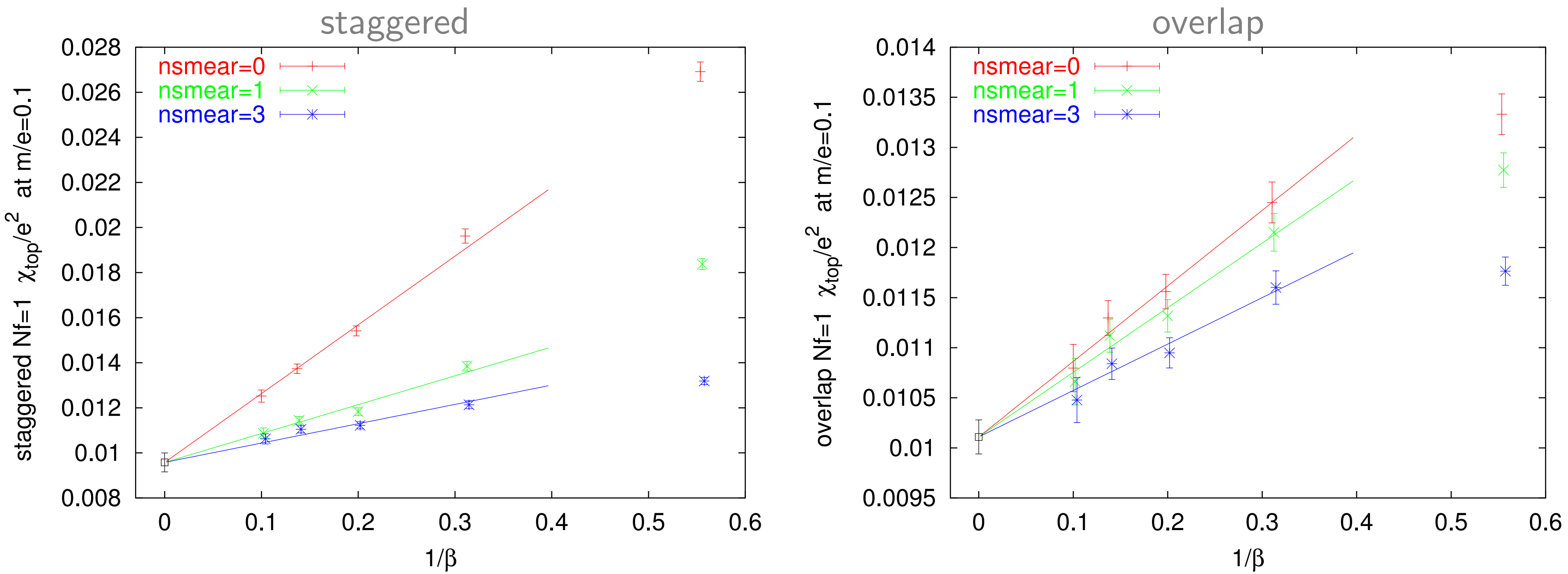


Overlap: small $O(a^2)$ effects, fairly insensitive to filtering (overlap yields good IR↔UV separation).

Staggered: large $O(a^2)$ effects, increase towards chiral limit, substantially reduced through filtering.

• $\chi_{\rm top}$ with $D_{\rm st}$ and $D_{\rm ov}$ in the continuum

Matched lattices: $\beta = 1.8, 3.2, 5.0, 7.2, 9.8$ with $L = 12, 16, 20, 24, 28$ yields $LM_{\eta'}\big|_{N_f=1, m=0} \simeq 5$.



⟶ Staggered extrapolation much steeper than with overlap (different scales).

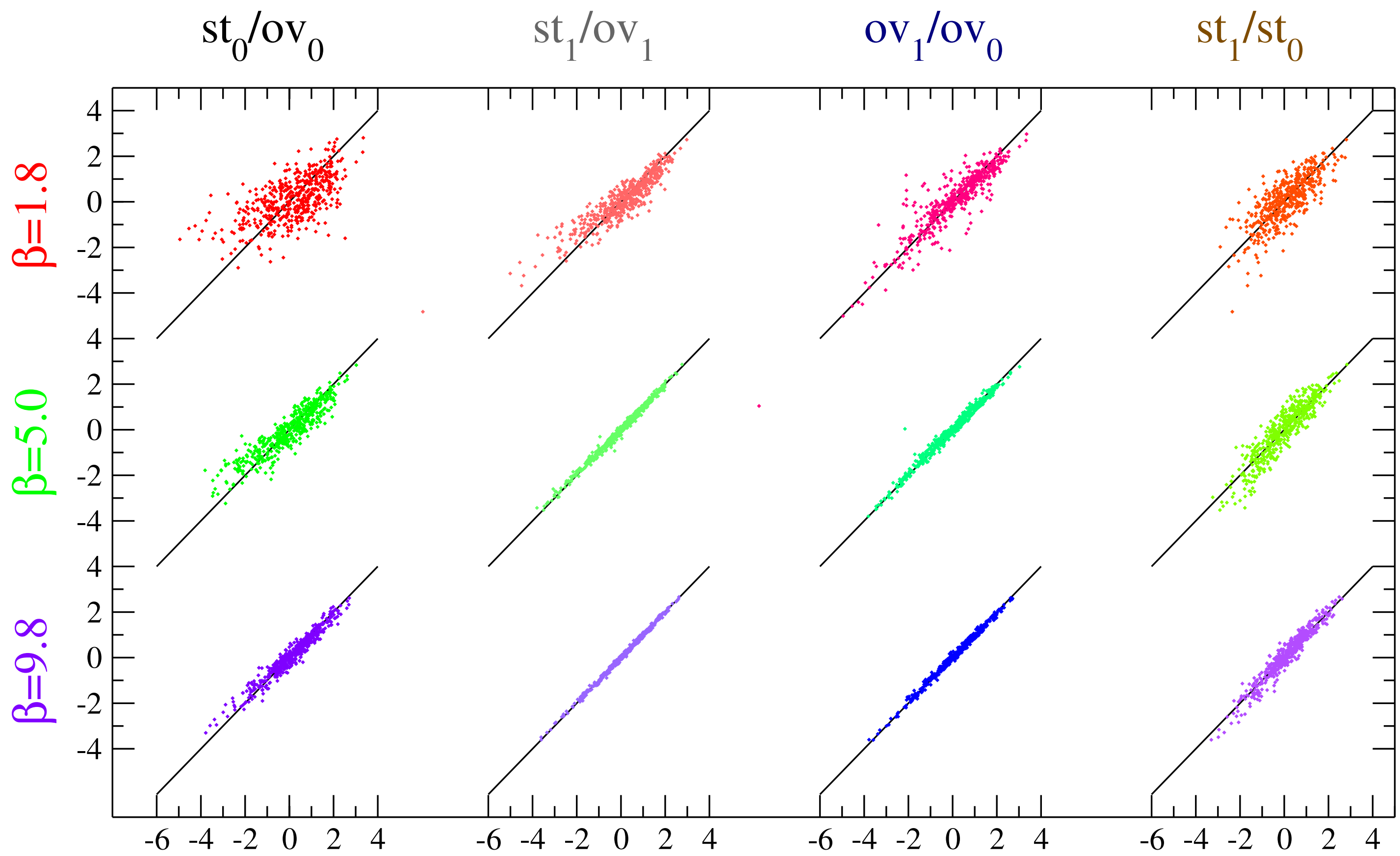
⟶ Combined fit with several filtering levels yields cost-effective continuum extrapolation.

⟹ Results for $\chi_{\rm top}^{N_f=1}$ suggest universal continuum limit, in spite of $\det^{1/2}(D_{\rm st})$.

⟶ Similar agreement in other continuum extrapolated quantities, e.g. for $F_{\rm HQ}$.

Correlation of $\frac{1}{2}\log\det(D_{\mathrm{st},m})$ and $\log\det(D_{\mathrm{ov},m})$ in 2D

Determinant ratio: $\frac{\lambda_1\,\lambda_2\,\ldots}{\lambda'_1\,\lambda'_2\,\ldots}\Big|_{\mathrm{ov}} \simeq \frac{\gamma_1\,\gamma_2\,\ldots}{\gamma'_1\,\gamma'_2\,\ldots}\Big|_{\mathrm{st}}$? ($\gamma_k = \sqrt{\lambda_{2k-1}\lambda_{2k}}$ geometric staggered mean)



⟶ At fixed quark mass $\det^{1/2}(D^{\mathrm{1APE}}_{\mathrm{st},m})$ in 2D generates ensemble that is <u>closer</u> to the one from $\det(D^{\mathrm{1APE}}_{\mathrm{ov},m})$ than the latter would be to $\det(D^{\mathrm{none}}_{\mathrm{ov},m})$, and the agreement improves with β.

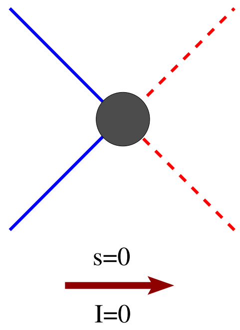
⟹ Maybe, $D_{\mathrm{ov},m}$ is a "candidate" operator with $\det(D_{\mathrm{ca},m}) = \mathrm{const}\cdot\det^{1/2}(D_{\mathrm{st},m})(1+O(a^2))$.

Low-energy unitarity and SXPT

- **continuum chiral perturbation theory (XPT)**

Observation: In QCD with $p \ll 1\,\mathrm{GeV}$ chiral symmetry constrains interactions of low-energy degrees of freedom with each other and with heavier particles (e.g. nuclei).

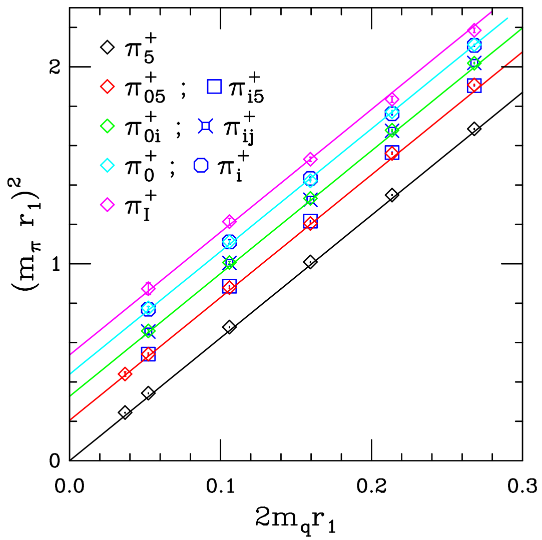
Consider $\pi\pi$ forward scattering $(s=0, I=0)$ at low momentum:



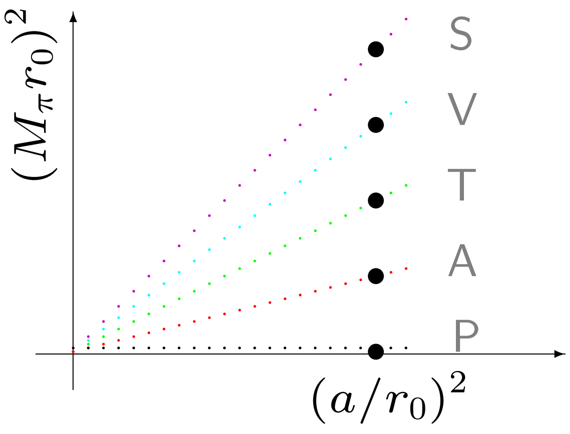
$$F_{\pi\pi}(\nu) = T^{I=0}[0, 2M_\pi(M_\pi+\nu), 2M_\pi(M_\pi-\nu)]$$
$$= -\frac{M_\pi^2}{F_\pi^2} + O(p^4) \quad \xrightarrow{M_\pi, p \downarrow 0} \quad 0$$

J.Gasser, H.Leutwyler, Ann.Phys. 158, 142 (1984), NPB, 250, 465 (1985)

- **staggered chiral perturbation theory (SXPT)**



Taste splitting makes most $\bar{d}(\gamma_5 \otimes T)u$ combinations become non-Goldstone bosons:



W.J.Lee, S.R.Sharpe, PRD 60, 114503 (1999) [hep-lat/9905023]

Assumption: With N_f flavors of (4-taste) quark fields the pattern of SSB is $SU(4N_f)_L \times SU(4N_f)_R \;\rightarrow\; SU(4N_f)_V$ leading to $16N_f{}^2 - 1$ pseudo-Goldstone bosons, collected in the 12×12 matrix ($N_f = 3$)

$$U = e^{\mathrm{i}\Phi/f} \qquad \Phi = \begin{pmatrix} \Phi_u & \pi^+ & K^+ \\ \pi^- & \Phi_d & K^0 \\ K^- & \bar{K}^0 & \Phi_s \end{pmatrix} = \sum_{a,b=1}^{9,16} \Phi^{ab} \frac{\lambda^a}{2} T^b \qquad M = \begin{pmatrix} m_u I_4 & 0 & 0 \\ 0 & m_d I_4 & 0 \\ 0 & 0 & m_s I_4 \end{pmatrix}$$

that transforms as $U \rightarrow V_L U V_R^\dagger$ under chiral rotations with unitary V_L, V_R.

With $f \simeq 122\,\mathrm{MeV}$ and $\Sigma \simeq (270\,\mathrm{MeV})^3$ the LO-Lagrangian (counting scheme $p^2 \sim m \sim a^2$) reads

$$L = \frac{f^2}{8}\mathrm{tr}(\partial_\mu U \partial_\mu U^\dagger) - \frac{\Sigma}{2}\mathrm{tr}(MU + MU^\dagger) + \frac{2m_0^2}{3}(\Phi_{u,\mathrm{TS}} + \Phi_{d,\mathrm{TS}} + \Phi_{s,\mathrm{TS}}) + a^2 V_{\mathrm{TB}}$$

C.Aubin, C.Bernard, PRD 68, 034014 (2003) [hep-lat/0304014] & PRD 68, 074011 (2003) [hep-lat/0306026]

and allows for systematic treatment of quantities covered by XPT, e.g. M_π, f_π, M_K, f_K .

- ⊕ SXPT analysis includes taste breaking effects.
- ⊕ Overall fits with horrific covariance matrices ("fitting herds of elephants") yield acceptable χ^2/d.o.f.
- ⊕ Some tests [$N_t = 1.28(12)$ per flavor, SXPT logs] successful, more [e.g. Sharpe, van de Water] to come.
- ⊖ What about physical observables not covered by (S)XPT ?
- ⊖ Unphysical tastes excised from predictions, but differently in valence and sea sector (unitarity?).

Summary

- Full QCD with $N_f = 2+1$ staggered fermions is controversial, since the Boltzmann weight $\det^{1/4}(D_{\mathrm{st}})$ assumes a taste symmetry which is only approximate.
- Formally, the taste symmetry breaking is due to a dimension 5 Wilson-type term in the taste basis and should thus go away in the continuum limit.
- Weak coupling, filtering, RG blocking reduce the taste splitting and give staggered quarks more appealing features, but there is no guarantee that no trace is left[1] in the continuum.
- One legal 1-flavor D_{ca} with $\det(D_{\mathrm{ca}}) = \mathrm{const}\cdot\det^{1/4}(D_{\mathrm{st}})(1+O(a^2))$ is sufficient to re-interpret existing MILC ensembles as being generated with a **local** action.
- The problem of (exact) **unitarity** in the fundamental theory remains, unless same D_{ca} is used in valence sector, too. Otherwise "partially quenched" situation with unitarity restored with $a \to 0$?

[1] Caveat: at least in some theories the cases $m=0$ and $m>0$ might be different in this respect.